

Generalized Derivations of Lie Algebras

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Suppose L is a finite-dimensional Lie algebra with multiplication $\mu: L \wedge L \rightarrow L$. Let $\Delta(L)$ denote the set of triples (f, f', f'') , with $f, f', f'' \in \text{Hom}(L, L)$, such that $\mu \circ (f \wedge I_L + I_L \wedge f') = f'' \circ \mu$. We consider the Lie algebra $\text{GenDer}(L) = \{f \in \text{Hom}(L, L) \mid \exists f', f'': (f, f', f'') \in \Delta(L)\}$. Well-researched subalgebras of $\text{GenDer}(L)$ include the derivation algebra, $\text{Der}(L) = \{f \in \text{Hom}(L, L) \mid (f, f, f) \in \Delta(L)\}$, and the centroid, $C(L) = \{f \in \text{Hom}(L, L) \mid (f, 0, f) \in \Delta(L)\}$. We now study the subalgebra $\text{QDer}(L) = \{f \in \text{Hom}(L, L) \mid \exists f': (f, f, f') \in \Delta(L)\}$, and the subspace $\text{QC}(L) = \{f \in \text{Hom}(L, L) \mid (f, -f, 0) \in \Delta(L)\}$. In characteristic $\neq 2$, $\text{GenDer}(L) = \text{QDer}(L) + \text{QC}(L)$ and we are concerned with the inclusions $\text{Der}(L) \subseteq \text{QDer}(L)$ and $C(L) \subseteq \text{QC}(L) \cap \text{QDer}(L)$. If $Z(L) = 0$ then $C(L) = \text{QC}(L) \cap \text{QDer}(L)$ and, under reasonable conditions on Lie algebras with toral Cartan subalgebras, we show $\text{QDer}(L) = \text{Der}(L) + C(L)$; if L is a parabolic subalgebra of a simple Lie algebra of rank > 1 in characteristic 0, then we even have $\text{GenDer}(L) = \text{ad}(L) + (I_L)$. In general $\text{QC}(L)$ is not closed under composition or Lie bracket; however, if $Z(L) = 0$ then $\text{QC}(L)$ is a commutative, associative algebra, and we describe conditions that force $\text{QC}(L) = C(L)$ or, equivalently, $\text{GenDer}(L) = \text{QDer}(L)$. We show that, in characteristic 0, $\text{GenDer}(L)$ preserves the radical of L , thus generalizing the classical result for $\text{Der}(L)$. We also discuss

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some applications of the main results to the study of functions $f \in \text{Hom}(L, L)$ such that $f \circ \mu$ or $\mu \circ (f \wedge I_L)$ defines a Lie multiplication. © 2000 Academic Press

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1. INTRODUCTION

Let A be a nonassociative algebra over a field of characteristic $\neq 2$ with multiplication $(a, b) \mapsto \mu(a, b)$. A *derivation* of A is an element $f \in \text{Hom}(A, A)$ such that

$$\mu(f(x), y) + \mu(x, f(y)) = f(\mu(x, y))$$

for all $x, y \in A$. The set $\text{Der}(A)$ of derivations of A is a well-studied Lie subalgebra of $\mathfrak{gl}(A)$. In this paper, we investigate a natural generalization of derivations.

We call $f \in \text{Hom}(A, A)$ a *generalized derivation* of A if there exist elements $f', f'' \in \text{Hom}(A, A)$ such that,

$$\mu(f(x), y) + \mu(x, f'(y)) = f''(\mu(x, y)) \quad (1.1)$$

for all $x, y \in A$, and we call $f \in \text{Hom}(A, A)$ a *quasiderivation* of A if there exists $f' \in \text{Hom}(A, A)$ such that

$$\mu(f(x), y) + \mu(x, f(y)) = f'(\mu(x, y))$$

for all $x, y \in A$. The set of generalized derivations of A forms a Lie subalgebra of $\mathfrak{gl}(A)$ which we denote by $\text{GenDer}(A)$. The Lie subalgebra consisting of quasiderivations is then denoted by $\text{QDer}(A)$.

Another important subalgebra of $\text{GenDer}(A)$ is the *centroid*, $C(A)$, of A consisting of $f \in \text{Hom}(A, A)$ such that

$$\mu(f(x), y) = \mu(x, f(y)) = f(\mu(x, y))$$

for all $x, y \in A$. The centroid is even closed under composition. It is easy to show that, for Lie algebras with trivial center, $C(L)$ is commutative (see, e.g., [14]) and so is the largest commutative subring of $\text{Hom}(L, L)$, containing the base field, over which L is an algebra.

For centerless Lie algebras L , we have the tower

$$L \simeq \text{ad}(L) \subseteq \text{Der}(L) \subseteq \text{QDer}(L) \subseteq \text{GenDer}(L) \subseteq \mathfrak{gl}(L), \quad (1.2)$$

where $\text{ad}(L)$ is the algebra of inner derivations. We are particularly interested in conditions that guarantee the collapsing of some of these inequalities. There has been much study of the inclusion $\text{ad}(L) \subseteq \text{Der}(L)$ from

this viewpoint (see, e.g., [5, 15] and the bibliographies therein). The next inclusion of Eq. (1.2) is always strict, for we have

$$\text{QDer}(L) \supseteq \text{Der}(L) \oplus C(L) \quad (\text{vector space direct sum}) \quad (1.3)$$

and $0 \neq I_L \in C(L)$. The interesting pursuit is for conditions that force $\text{Der}(L) + C(L) = \text{QDer}(L)$.

Also of interest is the subset of $\text{GenDer}(A)$ consisting of $f \in \text{Hom}(A, A)$ such that

$$\mu(f(x), y) = \mu(x, f(y))$$

for all $x, y \in A$ and we call this set the *quasicentroid* of A , denoting it by $\text{QC}(A)$. It is easy to verify that

$$C(A) \subseteq \text{QDer}(A) \cap \text{QC}(A), \quad (1.4)$$

and, for skew-commutative or commutative A , that

$$\text{GenDer}(A) = \text{QDer}(A) + \text{QC}(A) \quad (1.5)$$

(see Proposition 3.3). In his study of Levi factors in derivation algebras of nilpotent Lie algebras, Benoist [1] required the fact that $\text{GenDer}(S) = \text{QDer}(S) + C(S)$ when S is a semisimple Lie algebra in characteristic 0. He established this by showing $\text{QC}(S) = C(S)$ for such algebras. This equality does not hold for general Lie algebras. Indeed, the quasicentroid need not be closed under composition or Lie multiplication. This motivates a study of the structure of $\text{QC}(L)$ with particular reference to conditions that guarantee equality in the inclusion,

$$C(L) \subseteq \text{QC}(L). \quad (1.6)$$

We assume throughout this paper that algebras are finite-dimensional.

In Section 2, we recall some definitions and introduce notation.

Section 3 contains elementary observations about generalized derivations, some of which are technical results to be used later. The section also includes a characterization of Lie algebras L for which $\text{GenDer}(L) = \mathfrak{gl}(L)$. For such algebras we even have $\text{QDer}(L) = \mathfrak{gl}(L)$; this includes three-dimensional simple algebras.

In Section 4, we examine the structure of $\text{QDer}(L)$ for centerless Lie algebras with a toral Cartan subalgebra T . We first study $\text{QDer}(L)_0$, which consists of those quasiderivations f for which $T \cdot f = 0$. For example, if L is solvable, then $\text{QDer}(L)_0 \subseteq \text{Der}(L) + C(L)$. In general, $\text{QDer}(L) \neq$

$\text{Der}(L) + C(L)$, as is seen when L is three-dimensional simple. That example, however, characterizes the obstruction to equality for parabolic subalgebras of semisimple Lie algebras in characteristic 0; namely, $\text{QDer}(L) = \text{Der}(L) + C(L)$ if and only if the summands in the split form have rank > 1 . More generally, we go on to show that equality holds if the algebra is generated by *special* weight spaces (see Section 2 for definition).

Section 5 has several results dealing with $\text{QC}(L)$ for centerless Lie algebras L . For such algebras, we show that $\text{QC}(L)$ is closed under composition and is, in fact, commutative. Our analysis makes strong use of the Fitting decomposition of $\text{QC}(L)$ with respect to a Cartan subalgebra H of L : the Fitting 0-component is precisely $C(L)$ while the Fitting 1-component maps L into $Z(H)$ and maps H to 0. We show that $\text{QC}(L) = C(L) \oplus A$ in which A is an (associative-algebra) ideal and $A^2 = 0$. The information obtained on the structure of $\text{QC}(L)$ also reveals conditions that force equality in Eq. (1.6). For example, equality holds if $\text{ad}_L(H)$ does not contain elements that are nilpotent of index 2 (in particular, when H is a torus), or if $L = [L, L]$. We show also that equality holds in Eq. (1.4) for centerless Lie algebras. Thus, by Eq. (1.5) the algebras for which $\text{QC}(L) = C(L)$ are precisely those for which $\text{QDer}(L) = \text{GenDer}(L)$. Semisimple elements of $\text{QC}(L)$ necessarily lie in $C(L)$, so it is especially interesting to study the nilpotent elements of $\text{QC}(L)$ and we show that these always map L into the nilradical; in fact, with some restrictions on the field, we show that, if $f \in \text{QC}(L)$ is nilpotent of index m , then $f(L)$ generates a nilpotent ideal of index m .

Section 6 is devoted to showing that, in characteristic 0, generalized derivations of L preserve the solvable radical of L , thus generalizing the classical result for derivations.

In Section 7, we deal with applications of our analyses to two natural actions of $\mathfrak{gl}(V)$ on Lie algebra structures on V . Our study of $\text{QC}(L)$ augments results of Ikeda on *projective doubles* of Lie algebras (L, μ) , i.e., Lie algebras (L, ρ) with $\text{ad}(L, \rho) \subseteq \text{Der}(L, \mu)$. In particular, we look at *inner* projective doubles, wherein $\rho(x, y) = \mu(f(x), y)$ for some $f \in \text{Hom}(L, L)$: for centerless L , $f \in \text{Hom}(L, L)$ induces an inner projective double if and only if $f \in \text{QC}(L)$ with $f^2 \in C(L)$. We study also an analogous question of when $f \circ \mu$ is a Lie multiplication for nonsingular $f \in \text{Hom}(L, L)$. This is always the case if $f \in C(L)$, and we call (L, μ) *robust* if $f \circ \mu$ is a Lie multiplication *only* when $f \in C(L)$. We give a cohomological condition for robustness that is analogous to that for classical rigidity, though we point out the independence of the concepts. Our study of $\text{QDer}(L)$ is relevant here and we show, for example, that parabolic subalgebras of split simple algebras of rank > 1 are robust.

The paper is interspersed with examples that emphasize the sharpness of the results.

We note that the term *generalized derivation* has been used with various technical meanings different from ours (see, e.g., [6, 7, 21]).

2. DEFINITIONS AND NOTATION

We assume throughout that the characteristic of the ground field κ is not 2.

We shall be concerned chiefly with finite-dimensional Lie algebras, though some elementary results are valid for more general algebras. Let (A, μ) be a nonassociative algebra over κ . Thus, A is a vector space over κ , and $\mu: A \times A \rightarrow A$ is a bilinear map. It is also convenient at times to view the multiplication as a linear map $\mu: A \otimes A \rightarrow A$. We often write A in place of (A, μ) . For $n \geq 1$, we define A^n to be the span of all products $x_1 \cdots x_n$, with $x_i \in A$, no matter how associated. The (two-sided) ideal of A generated by a set S will be denoted by $\langle S \rangle$. The *annihilator* of A is

$$\text{Ann}(A, \mu) = \{x \in A \mid \mu(x, y) = \mu(y, x) = 0, \quad \forall y \in A\}.$$

It is sometimes convenient to deal with the set of triples (f, f', f'') satisfying Eq. (1.1) and we denote this collection by $\Delta(A)$. Thus, $\Delta(A)$ is a Lie subalgebra of $\mathfrak{gl}(A) \times \mathfrak{gl}(A) \times \mathfrak{gl}(A)$.

If (A, μ) is a nonassociative algebra over κ and R is a commutative algebra over κ , we denote by $A[R]$ the algebra over κ whose underlying vector space is $A \otimes R$ with multiplication satisfying

$$(a_1 \otimes r_1) \cdot (a_2 \otimes r_2) = \mu(a_1, a_2) \otimes r_1 r_2,$$

for $a_1, a_2 \in A, r_1, r_2 \in R$.

Let L be a Lie algebra over κ . For $X, Y \subseteq L$, L a Lie algebra, let $Z_X(Y)$ denote the *centralizer* of Y in X , i.e., $\{x \in X \mid [x, Y] = 0\}$; as usual, $Z(L) = Z_L(L)$ is the *center* of L ; if $Z(L) = 0$, we may say that L is *centerless*. If M and N are L -modules, the *transporter* of M to N is the ideal $\text{Tr}(M, N) = \{x \in L \mid x \cdot M \subseteq N\}$; of special interest is the case when M and N are ideals in L and therefore modules with respect to the adjoint action. We denote by $\text{Rad}(L)$ the (solvable) radical of L and by $\text{NilRad}(L)$ the maximum nilpotent ideal of L . For $S \subseteq L$, $\langle S \rangle$ denotes the ideal generated by S . We say L is *directly indecomposable* if $L = H \oplus K$, a direct sum of ideals, implies H or K is 0.

An element, x , of a Lie algebra, L , is *semisimple* if $\text{ad}(x)$ is a semisimple (diagonalizable over an extension field) element of $\text{Hom}(L, L)$. A *torus*, T , is an abelian subalgebra of a Lie algebra, L , which consists of semisimple elements. We say T is *split* if, for $x \in T$, the characteristic roots of $\text{ad}(x)$ belong to the base field. Unless otherwise indicated, the tori we consider

will be split. We consider several naturally induced L -modules on which tori in L act diagonally; if M is such an L -module then, for $\alpha \in T^*$, α is a *weight* of T on M if the *weight space*, $M_\alpha = \{m \in M \mid t \cdot m = \alpha(t)m, \forall t \in T\} \neq 0$. We denote by $\mathcal{W}(M)$ the weights of T in M (T will always be clear in context).

For $\alpha \in \mathcal{W}(T)$, we set $L_{(\alpha)} = \bigoplus_{c \in \kappa} L_{c\alpha}$ and $L_{(\alpha)'} = \bigoplus_{\beta \notin \kappa \alpha} L_\beta$. We say a weight α for T in L is *special* if $L_{(\alpha)}$ does not contain a nontrivial ideal of L .

A *Heisenberg Lie algebra* of dimension $2n + 1$ has basis $\{x_i\}$ with $1 \leq i \leq 2n + 1$ and nonzero products, $[x_i, x_{n+i}] = -[x_{n+i}, x_i] = x_{2n+1}$ for $i = 1, \dots, n$.

If L is a Lie algebra and M an L -module, $Z^i(L, M)$, $B^i(L, M)$, and $H^i(L, M)$ denote the i th-cocycles, coboundaries, and cohomology of L with coefficients in M , while δ denotes the coboundary operator. For more details, see [[9], p. 93 et seq]. We define \check{L} to be the trivial L -module on the underlying vector space of L , while the adjoint module is still denoted by L . To distinguish the coboundary map δ on the complex $C(L, L)$, we let $\check{\delta}$ denote the coboundary map on $C(L, \check{L})$. Observe that $B^2(L, \check{L}) = \text{Hom}(L, L) \circ \mu$. Also, if $L = (L, \mu)$, $f \in \text{Hom}(L, L)$, then $\delta(f \circ \mu) = \check{\delta}(f \circ \mu)$.

Let V be a vector space. The identity transformation of V is denoted I_V . For $S \subseteq V$, $\langle S \rangle$ denotes the linear span of S . We denote by $V \otimes V$ the symmetric product, and by $V \wedge V$ the exterior product; we may view these as subspaces of $V \otimes V$, where $V \otimes V = (v \otimes w + w \otimes v \mid v, w \in V)$ and $V \wedge V = (v \otimes w - w \otimes v \mid v, w \in V)$.

3. GENERAL REMARKS

For any nonassociative algebra A , $\Delta(A)$ is an algebraic Lie algebra and so the semisimple and nilpotent components of generalized derivations are generalized derivations.

It is easy to see that generalized derivations preserve the annihilator of a nonassociative algebra, and hence the center of a Lie algebra. However, although $\text{Der}(L)$ and $C(L)$ clearly preserve the derived algebra of a Lie algebra L , neither $\text{QDer}(L)$ nor $\text{QC}(L)$ need do so: for quasiderivations, this is evident in the two-dimensional, nonabelian Lie algebra spanned by x, y with $[x, y] = y$ (the linear map satisfying $x \mapsto 0$, $y \mapsto x$ is a quasiderivation); see Example 5.7 for a Lie algebra L and $f_3 \in \text{QC}(L)$ such that $f_3([L, L]) \not\subseteq [L, L]$.

LEMMA 3.1. *Let A be a nonassociative algebra. Then*

- (1) $\text{QDer}(A)$ is a Lie subalgebra of $\text{GenDer}(A)$.

- (2) $[\text{Der}(A), C(A)] \subseteq C(A)$.
- (3) $[\text{QDer}(A), \text{QC}(A)] \subseteq \text{QC}(A)$.
- (4) $C(A) \subseteq \text{QDer}(A)$.
- (5) $[\text{QC}(A), \text{QC}(A)] \subseteq \text{QDer}(A)$.

Proof. (1), (2), (3) are immediate. For (4), observe that $f \in C(A)$ implies $(f, f, 2f) \in \Delta(A)$. For (5), $f, g \in \text{QC}(L)$ implies $([f, g], [f, g], 0) \in \Delta(A)$. ■

Remark 3.2. If A is commutative or skew commutative, then $(f, f', f'') \in \Delta(A)$ implies $(f', f, f'') \in \Delta(A)$.

PROPOSITION 3.3. *Let A be a commutative or skew-commutative algebra, then*

- (1) $\text{GenDer}(A) = \text{QDer}(A) + \text{QC}(A)$.
- (2) $\text{QC}(A) + [\text{QC}(A), \text{QC}(A)]$ is an ideal in the Lie algebra $\text{GenDer}(A)$.

Proof. For (1), we note that $(g, g', g'') \in \Delta(A)$ implies

$$(g, g', g'') = (f, f, g'') + (e, -e, 0),$$

where $f = (g + g')/2$, and $e = (g - g')/2$. With this, (2) follows from (3) and (5) of Lemma 3.1. ■

Suppose $A = H \oplus K$, a direct sum of ideals, and $\text{Ann}(A) = 0$. Then, if $f \in \text{GenDer}(A)$, f must preserve H and K . Thus, we generalize the well known result for derivations to obtain the following useful result.

LEMMA 3.4. *If $A = H \oplus K$ is the direct sum of ideals, and $\text{Ann}(A) = 0$, then*

- (1) $\text{GenDer}(A) = \text{GenDer}(H) \oplus \text{GenDer}(K)$.
- (2) $\text{QDer}(A) = \text{QDer}(H) \oplus \text{QDer}(K)$.
- (3) $C(A) = C(H) \oplus C(K)$.
- (4) $\text{QC}(A) = \text{QC}(H) \oplus \text{QC}(K)$.

The quasiderivations of an algebra can be embedded as derivations in a larger algebra. (That is their key role in [1].) For this, let A be a nonassociative algebra over κ and t an indeterminant. We define an algebra, \check{A} , over κ , canonically associated to A , as $\check{A} = A[t\kappa[t]/(t^3)]$. We write at (at^2) in place of $a \otimes t$ ($a \otimes t^2$). If U is a subspace of A such that $A = U \oplus A^2$ then

$$\check{A} = At + At^2 = At + A^2t^2 + Ut^2.$$

Now we define a map $\iota_U: \text{QDer}(A) \rightarrow \text{Der}(\check{A})$ so that, for $(f, f, f') \in \Delta(A)$,

$$\iota_U(f)(at + bt^2 + ut^2) = f(a)t + f'(b)t^2$$

for all $a \in A$, $b \in A^2$ and $u \in U$. Note that ι_U is injective and that $\iota_U(f)$ does not depend on the choice of f' .

Now, for any nonassociative algebra B , let

$$\text{ZDer}(B) = \{f \in \text{Hom}(B, B) \mid f(B) \subseteq \text{Ann}(B), f(B^2) = 0\}.$$

Then $\text{ZDer}(B)$ is an ideal in $\text{Der}(B)$.

The following proposition is implicit in [13, p.191].

PROPOSITION 3.5. *If A is a nonassociative algebra with $\text{Ann}(A) = 0$, and ι_U as above, then $\text{Der}(\check{A})$ is a semidirect sum:*

$$\text{Der}(\check{A}) = \iota_U(\text{QDer}(A)) \oplus \text{ZDer}(\check{A}).$$

Proof. Observe that $\text{Ann}(\check{A}) = At^2$. So any linear $f: At + Ut^2 \rightarrow At^2$ extends to an element of $\text{ZDer}(\check{A})$ by taking $f(A^2t^2) = 0$. Thus, given any $g \in \text{Der}(\check{A})$, we can subtract an element of $f \in \text{ZDer}(\check{A})$ and thereby force $(g - f)(At) \subseteq At$ and $(g - f)(Ut^2) = 0$ (note that $g(Ut^2) \subseteq g(\text{Ann}(\check{A})) \subseteq \text{Ann}(\check{A})$). Also, since $A^2 = A^2t^2$, $(g - f)(A^2t^2) \subseteq A^2t^2$. Thus $(h, h, h') \in \Delta(A)$, where $(g - f)(at) = h(a)t$, $(g - f)(bt^2) = h'(b)t$ for $a \in A$, $b \in A^2$, and so, $(g - f) = \iota_U(h) \in \iota_U(\text{QDer}(A))$. ■

Observe that, in the situation of Proposition 3.5, $\iota_U(\text{QDer}(A))$ may also be viewed as the natural image of $\text{Der}(\check{A})$ in $\text{Hom}(A/A^2, \check{A}/A^2)$.

If (A, μ) is an algebra and f in $\text{Hom}(A, A)$, we define $f^\# \in \text{Hom}(A \otimes A, A \otimes A)$ so that $f^\#(v \otimes w) = f(v) \otimes w + v \otimes f(w)$ for all $v, w \in A$. Note that $\mu \circ f^\#$ factors through $\mu(A \otimes A)$ (i.e., $\mu \circ f^\# = f' \circ \mu$), for some $f' \in \text{Hom}(A, A)$, if and only if $f^\#(\text{Ker}(\mu)) \subseteq \text{Ker}(\mu)$.

LEMMA 3.6. *Let (A, μ) be an algebra and $f \in \text{Hom}(A, A)$. Then $f \in \text{QDer}(A, \mu)$ if and only if $f^\#(\text{Ker}(\mu)) \subseteq \text{Ker}(\mu)$.*

Proof. $f \in \text{QDer}(A)$ if and only if there exists $f' \in \text{Hom}(A, A)$ such that $f' \circ \mu = \mu \circ f^\#$. ■

PROPOSITION 3.7. *If (A, μ) is an algebra with $\mu \neq 0$, and $\text{QDer}(A, \mu) = \text{gl}(A)$, then*

$$(1) \quad \dim(A) \leq 3.$$

(2) *If $\dim(A) = 1$, then $(A, \mu) \cong \kappa$. If $\dim(A) = 2$, then A is a solvable Lie algebra, and if $\dim(A) = 3$, then (A, μ) is a simple, skew-commutative, nonassociative algebra.*

Conversely, if L is one of these algebras, $\text{QDer}((A, \mu)) = \mathfrak{gl}(A)$.

Proof. $\mathfrak{gl}(A)$ acts on $A \otimes A$ via $f \cdot (v \otimes w) = f^\#(v \otimes w)$ for all $v, w \in A$ with $f^\#$ as above. By Lemma 3.6, $\mathfrak{gl}(A) = \text{QDer}(A)$ implies $\mathfrak{gl}(A) \cdot \text{Ker}(\mu) \subseteq \text{Ker}(\mu)$. Now the only proper subspaces of $A \otimes A$, invariant under this action of $\mathfrak{gl}(A)$, are $A \wedge A$ and $A \otimes A$. Thus we have $\mu: A \otimes A \rightarrow A$ with kernel 0, $A \wedge A$, or $A \otimes A$. Using $\dim(A) \geq \dim(A \otimes A) - \dim(\text{Ker}(\mu))$, an easy computation shows that if $\text{Ker}(\mu) = 0$ or $A \wedge A$, then $\dim(A) \leq 1$, while if $\text{Ker}(\mu) = A \otimes A$, $\dim(A) \leq 3$.

If $\dim(A) = 1$, then $A \wedge A = 0$, so that the only possibility for $\text{Ker}(\mu)$ is 0, and the only nontrivial, one-dimensional algebra over a field is isomorphic to that field. If $\dim(A) = 2$ or 3, then the only possibility for $\text{Ker}(\mu)$ is $A \otimes A$, so μ must be skew commutative. If $\dim(A) = 2$, then $\dim(\mu(A \otimes A)) = 1$, so that A is the two-dimensional nonabelian Lie algebra. If $\dim(A) = 3$, then $\dim(\mu(A \otimes A)) = 3$, so μ must be surjective and it follows that A cannot have a proper ideal.

Noting the $\mathfrak{gl}(A)$ -invariance of the kernel of μ for these algebras, the converse is straightforward. ■

The next theorem makes use of Theorem 5.11 (whose proof is independent of this result).

THEOREM 3.8. *Let L be a Lie algebra such that $\text{GenDer}(L) = \mathfrak{gl}(L)$. Then $\text{QDer}(L) = \mathfrak{gl}(L)$. Thus, either L is abelian, two-dimensional solvable or three-dimensional simple. Conversely, if L is one of these algebras, $\text{GenDer}(L) = \mathfrak{gl}(L)$.*

Proof. Assume $\text{GenDer}(L) = \mathfrak{gl}(L)$ and that $[L, L] \neq 0$.

Claim: $\text{QC}(L) = (I_L)$. To see this, let $K = \text{QC}(L) + [\text{QC}(L), \text{QC}(L)]$, the ideal of $\text{GenDer}(L)$ generated by $\text{QC}(L)$ (Proposition 3.3). Then K is an ideal of $\text{GenDer}(L) = \mathfrak{gl}(L)$ containing I_L .

Case 1: $Z(L) \neq 0$. Since $Z(L) \neq L$ and K stabilizes $Z(L)$, we get that $K = (I_L)$ since (I_L) is the only ideal of $\mathfrak{gl}(L)$ that stabilizes a proper subspace of L .

Case 2: $Z(L) = 0$. By Theorem 5.11, $\text{QC}(L)$ is commutative and therefore K is an abelian ideal in $\mathfrak{gl}(L)$. It follows that $K = (I_L)$.

By the above claim and Proposition 3.1, $\text{QDer}(L) + (I_L) = \mathfrak{gl}(L)$. But, since $(I_L, I_L, 2I_L) \in \Delta(L)$, we know $(I_L) \in \text{QDer}(L)$. Therefore, $\text{QDer}(L) = \mathfrak{gl}(L)$. The converse follows at once from Proposition 3.7. ■

The following easy lemma provides a useful connection between quasiderivations and Lie algebra cohomology.

LEMMA 3.9. *Let L be a Lie algebra. For $f, f' \in \text{Hom}(L, L)$,*

(1) $f \in \text{QDer}(L)$ if and only if $\delta f \in B^2(L, \mathring{L})$. More specifically, $(f, f, f') \in \Delta(L)$ if and only if $\delta f = \mathring{\delta}(f' - f)$.

(2) If $\mathring{\delta}f \in Z^2(L, L)$, then, $\forall x, y, z \in L$,

$$[x, f([y, z])] + [y, f([z, x])] + [z, f([x, y])] = 0.$$

In particular, if $(f, f, f') \in \Delta(L)$, then

$$[x, (f' - f)([y, z])] + [y, (f' - f)([z, x])] + [z, (f' - f)([x, y])] = 0.$$

■

4. QUASIDERIVATIONS OF LIE ALGEBRAS

Let L be a Lie algebra containing a (diagonalized) torus T . Then

$$\text{QDer}(L) = \text{QDer}(L)_0 \oplus T \cdot \text{QDer}(L)$$

(the second summand is comprised of the nonzero weight spaces). We also take note of the subspace

$$\text{QDer}_T(L) = \{f \in \text{QDer}(L) \mid f(T) = 0\}.$$

LEMMA 4.1. *Let L be a Lie algebra and $T \subseteq L$ a torus. Then*

$$\text{QDer}(L) = \text{QDer}(L)_0 + \text{QDer}_T(L) + \text{ad}(L).$$

Proof. Let $f \in \text{QDer}(L)$. For $t, t' \in T$, $(t \cdot f)(t') = -[f(t), t']$ so that $t \cdot f + \text{ad}(f(t)) \in \text{QDer}_T(L)$. Hence, $T \cdot \text{QDer}(L) \subseteq \text{QDer}_T(L) + \text{ad}(L)$.

■

4.1. The Structure of $\text{QDer}(L)_0$

We consider the structure of $\text{QDer}(L)_0$ for centerless Lie algebras.

LEMMA 4.2. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$, a torus with $T = L_0$, and $Z(L) = 0$. Then*

$$\text{QDer}(L)_0 = \text{QDer}_T(L)_0 + (I_L).$$

Proof. Let $f \in \text{QDer}(L)_0$ and suppose $(f, f, f') \in \Delta(L)$. Let B be a maximal independent set in $\mathcal{W}(L)$. Since $T \cap Z(L) = 0$, B is a basis of T^* and so there is a basis $\{t_\beta \mid \beta \in B\}$ of T dual to B . For each $\beta \in B$, let $0 \neq x_\beta \in L_\beta$. For $\beta, \gamma \in B$, we have

$$[f(t_\beta), x_\gamma] + [t_\beta, f(x_\gamma)] = f'([t_\beta, x_\gamma]).$$

Since $f \in \text{QDer}(L)_0$, we have that $f(T) \subseteq T$, and $f(L_\gamma) \subseteq L_\gamma$, so that

$$\gamma(f(t_\beta))x_\gamma + \gamma(t_\beta)f(x_\gamma) = \gamma(t_\beta)f'(x_\gamma).$$

Thus, if $\beta \neq \gamma$, then $\gamma(f(t_\beta)) = 0$. It follows that $f(t_\beta) = c_\beta t_\beta$ for some scalar c_β . In particular, f acts diagonally on T . Thus,

$$T = \bigoplus_{c \in \Gamma} T_c,$$

where Γ is the set of characteristic roots for the action of f on T , and T_c is the root space for c . It suffices now to prove that $|\Gamma| = 1$, for, if $T = T_c$, then

$$f - cI_L \in \text{QDer}_T(L)_0.$$

Claim (1): For any $0 \neq \alpha \in \mathcal{W}(L)$, there is a unique $c \in \Gamma$ such that $[T_c, L_\alpha] \neq 0$.

Proof. Suppose that $[T_{c_1}, L_\alpha] \neq 0$ and $[T_{c_2}, L_\alpha] \neq 0$, with $c_1, c_2 \in \Gamma$. Then there exists $t_i \in T_{c_i}$, $i = 1, 2$, such that $\alpha(t_i) = 1$. Let $0 \neq x_\alpha \in L_\alpha$. For $i = 1, 2$, we have

$$[f(t_i), x_\alpha] + [t_i, f(x_\alpha)] = f'([t_i, x_\alpha]),$$

or

$$c_i x_\alpha + f(x_\alpha) = f'(x_\alpha).$$

Hence, $c_1 = c_2$, proving Claim (1).

For each $c \in \Gamma$, let $w(c) := \{\alpha \in \mathcal{W}(L) \mid [T_c, L_\alpha] \neq 0\}$.

Claim (2): If $\alpha_i \in w(c_i)$, for $i = 1, 2$, with $c_1 \neq c_2$, then $[L_{\alpha_1}, L_{\alpha_2}] = 0$.

Proof. Take $t_i \in T_{c_i}$ such that $\alpha_i(t_i) = 1$. By Claim (1), we also know that $\alpha_1(t_2) = \alpha_2(t_1) = 0$. In particular, $\alpha_1 + \alpha_2 \neq 0$, so that if $[L_{\alpha_1}, L_{\alpha_2}] \neq 0$, we would have $[L_{\alpha_1}, L_{\alpha_2}] \not\subseteq T$. But then, $[t_i, [L_{\alpha_1}, L_{\alpha_2}]] = [L_{\alpha_1}, L_{\alpha_2}]$, for $i = 1, 2$. By Claim (1), this can only happen if $[L_{\alpha_1}, L_{\alpha_2}] = 0$, thus proving Claim (2).

Now, for each $c \in \Gamma$, let

$$H_c = T_c + \sum_{\alpha \in w(c)} L_\alpha.$$

Then we have $L = \bigoplus_{c \in \Gamma} H_c$, a direct sum as vector spaces. But, by the above, $H_c = Z_L(\sum_{c' \neq c} H_{c'})$ (equality holds since $Z(L) = 0$). Thus, each H_c is an ideal and $\bigoplus_{c \in \Gamma} H_c$ is an algebra direct sum. By the indecomposability of L , $|\Gamma| = 1$. ■

LEMMA 4.3. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$, a torus with $T = L_0$, $Z(L) = 0$, and $\dim(T) > 1$. Then, if $0 \neq t \in T$, there exist independent weights $\alpha, \beta \in \mathcal{W}(L)$ such that $\alpha(t) \neq 0$ and $\beta(t) \neq 0$.*

Proof. Fix $\alpha \in \mathcal{W}(L)$ for which $\alpha(t) \neq 0$ (α exists since $T \cap Z(L) = 0$) and assume $\beta(t) = 0$ for all $\beta \in \mathcal{W}(L)$, $\beta \notin (\alpha)$. Suppose $\beta \in \mathcal{W}(L) \setminus (\alpha)$ and $0 \neq \gamma \in \mathcal{W}(L) \cap (\alpha)$. Since $\beta + \gamma \notin (\alpha)$ but $\beta(t) + \gamma(t) = \gamma(t) \neq 0$, it follows that $\beta + \gamma \notin \mathcal{W}(L)$, so that $[L_\beta, L_\gamma] = 0$. Let

$$\widehat{T} = \{t' \in T \mid \alpha(t') = 0\},$$

so $T = (t) \oplus \widehat{T}$. Let

$$A = (t) + \sum_{0 \neq \beta \in (\alpha)} L_\beta,$$

and

$$B = \widehat{T} + \sum_{\beta \notin (\alpha)} L_\beta.$$

Clearly, A is a subalgebra of L , $L = A \oplus B$ as a vector space, and B centralizes A . Since $Z(L) = 0$, B must be precisely $Z_L(A)$ (the centralizer of A in L) so that B is a subalgebra of L and hence, both A and B are ideals. Thus, $L = A \oplus B$ is an algebra direct sum, contradicting the indecomposability of L . ■

LEMMA 4.4. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$ a torus with $T = L_0$, $Z(L) = 0$, and $\dim(T) > 1$. Then*

$$\text{QDer}_T(L)_0 \subseteq \text{Der}(L).$$

Proof. Let $f \in \text{QDer}_T(L)_0$. For $t \in T$, $x \in L$, $[t, f(x)] = f'([t, x])$. Hence, if $x \in L_\alpha$, $\alpha(t)f(x) = \alpha(t)f'(x)$. Thus, if $\alpha \neq 0$, $f(x) = f'(x)$, so f and f' agree on $[T, L]$.

It suffices to show that $f'(T \cap [L, L]) = 0$, whence $f'|_{[L, L]} = f|_{[L, L]}$, so that $f \in \text{Der}(L)$.

The space $T \cap [L, L]$ is spanned by products of the form $[x, y]$ where, for some $\alpha \neq 0$, $x \in L_\alpha$ and $y \in L_{-\alpha}$. Consider such x, y . By weights, $f'([x, y]) = [f(x), y] + [x, f(y)] \in L_0 = T$. Let now $z \in L_\beta$ for $\beta \neq \pm\alpha$. Using $f'([x, z]) = f([x, z])$ and $f'([y, z]) = f([y, z])$ and Lemma 3.9(2), we get $[z, (f' - f)([x, y])] = 0$, so that $[f'([x, y]), z] = [f([x, y]), z] = 0$. By Lemma 4.3, we must have $f'([x, y]) = 0$. ■

These lemmas give us the major result of this subsection.

THEOREM 4.5. *If L is a directly indecomposable Lie algebra, $T \subseteq L$ a torus with $T = L_0$, $Z(L) = 0$, and $\dim(T) > 1$, then*

$$\text{QDer}(L)_0 = \text{Der}(L)_0 + (I_L).$$

Proof. By Lemmas 4.1, 4.2, and 4.4, $\text{QDer}(L)_0 \subseteq \text{Der}(L)_0 \oplus (I_L)$. The reverse inclusion is clear. ■

COROLLARY 4.6. *Let L be a directly indecomposable Lie algebra with $L \neq [L, L]$, $T \subseteq L$ a torus with $T = L_0$, and $Z(L) = 0$. Then*

$$\text{QDer}(L)_0 = \text{Der}(L)_0 \oplus (I_L).$$

Proof. By Theorem 4.5, we may assume $T = (t)$. Since $L \neq [L, L]$, $L = (t) \oplus [t, L]$ (vector space direct sum), and $t \notin [L, L]$. Let $[t, L] = \sum_{\lambda \neq 0} L_\lambda$ with $[t, x] = \lambda x$ for $x \in L_\lambda$. Take $(f, f, f') \in \Delta(L)$ with $f \in \text{QDer}(L)_0$. By Lemma 4.2, we may assume $f(t) = 0$. Since f' may be arbitrarily redefined on a fixed complement of $[L, L]$, we may assume $f'(t) = 0$. Let $x \in L_\lambda$ with $\lambda \neq 0$. Since $f \in \text{QDer}(L)_0$, $f(x) \in L_\lambda$, so

$$\lambda f'(x) = f'([t, x]) = [t, f(x)] = \lambda f(x).$$

Hence, $f = f'$ and so $f \in \text{Der}(L)_0$. ■

COROLLARY 4.7. *Let L be a solvable Lie algebra, $T \subseteq L$ a (not necessarily diagonalized) torus with $T = L_0$, and $Z(L) = 0$. Then*

$$\text{QDer}(L)_0 = \text{Der}(L)_0 \oplus C(L),$$

and this is a direct sum as algebras. In fact, $C(L) = Z(\text{GenDer}(L))$.

Proof. We may assume that the base field is algebraically closed and, thus, that T is diagonalized. (Note that the 0-weight space is defined even if T is not diagonalized.) Let $L = \bigoplus_i L_i$ with each L_i directly indecomposable. Then T splits accordingly and, by Corollary 4.6, $\text{QDer}(L_i)_0 = \text{Der}(L_i)_0 \oplus (I_{L_i}) \subseteq \text{Der}(L)_0 \oplus Z_{C(L_i)}(\text{GenDer}(L_i))$. Thus, using Lemma 3.4,

$$\begin{aligned} \text{QDer}(L)_0 &\subseteq \text{Der}(L)_0 \oplus Z_{C(L)}(\text{GenDer}(L)) \\ &\subseteq \text{Der}(L)_0 \oplus C(L) \\ &\subseteq \text{QDer}(L)_0. \end{aligned}$$

Hence, we have equality throughout and $C(L) \subseteq Z(\text{GenDer}(L))$. But $Z(\text{GenDer}(L)) \subseteq Z_{\mathfrak{gl}(L)}(\text{ad}(L)) = C(L)$. ■

4.2. Consequences of Generation by Special Weight Spaces

Recall that a weight $\alpha \in {}^\circ\mathcal{W}(L)$ is called special if $L_{(\alpha)}$ does not contain a nontrivial ideal of L . Note that $[L_{(\alpha)}, L_{(\alpha')}] \subseteq L_{(\alpha')}$.

LEMMA 4.8. *If $T \subseteq L$ is a torus with $T = L_0$ and α is a special weight for T on L , then*

$$(1) \quad \dim(T) > 1.$$

(2) *If M is a direct summand of L such that $L_\alpha \cap M \neq 0$, then α is a special weight for $T \cap M$ in M .*

Proof. To see (1), we need only note that, if α is a special weight, then there must exist a weight which is linearly independent of α . Since an ideal generated by an element of a direct summand stays in the summand, (2) is clear. ■

LEMMA 4.9. *Let A, B be subspaces of a Lie algebra L such that $L = A + B$, $[A, B] \subseteq B$. If $x \in A$ is such that $[x, B] = 0$, then $[L^n, x] = [A^n, x]$ for all $n \geq 1$.*

Proof. We note first that $[B, [A^n, x]] = 0$ for all $n \geq 1$. This follows by induction using

$$\begin{aligned} [B, [A^n, x]] &= [B, [A, [A^{n-1}, x]]] \\ &= [[B, A], [A^{n-1}, x]] + [A, [B, [A^{n-1}, x]]] \\ &\subseteq [B, [A^{n-1}, x]] + [A, [B, [A^{n-1}, x]]]. \end{aligned}$$

From this, the lemma also follows by induction using

$$[L^n, x] = [A, [L^{n-1}, x]] + [B, [L^{n-1}, x]]. \quad \blacksquare$$

LEMMA 4.10. *Let $T \subseteq L$ be a torus and $\alpha \in \mathcal{W}(L)$. Then α is special if and only if, for all $0 \neq x \in L_{(\alpha)}$, $[x, L_{(\alpha)'}] \neq 0$.*

Proof. For $x \in L_{(\alpha)}$, $[x, L_{(\alpha)'}] \neq 0$ implies $[x, L_{(\alpha)}] \subseteq \langle x \rangle \cap L_{(\alpha)'} \neq 0$, so the sufficiency is clear.

For the necessity, put $A = L_{(\alpha)}$, $B = L_{(\alpha)'}$ in Lemma 4.9. If $0 \neq x \in A$, with $[x, B] = 0$, then $\langle x \rangle = (x) + \sum_n [L_{(\alpha)}^n, x] \subseteq L_{(\alpha)}$, whence α is not special. ■

LEMMA 4.11. *Let L be a Lie algebra, and $T \subseteq L$ a torus with $T = L_0$. Suppose that $Z(L) = 0$ and that L is generated by special weight spaces. Then, for any $\alpha \neq 0$,*

$$\text{QDer}_T(L)_\alpha = 0.$$

Equivalently, $T \cdot \text{QDer}_T(L) = 0$.

Proof. By Lemma 4.8, if L is generated by special weight spaces, then this is true for any direct summand of L . Thus, we may assume that L is indecomposable. Now take $f \in \text{QDer}_T(L)_\alpha$ with $\alpha \neq 0$. Let $t \in T$, $0 \neq \beta \in \mathcal{W}(L)$, and $0 \neq x_\beta \in L_\beta$. Since $f(T) = 0$, $[t, f(x_\beta)] = f'([t, x_\beta])$ so we have

$$(\alpha(t) + \beta(t))f(x_\beta) = \beta(t)f'(x_\beta). \quad (4.1)$$

Thus, if $f(x_\beta) \neq 0$, α is a multiple of β . Also, for $\beta \neq 0$, $- \alpha$, $f(x_\beta) \neq 0$ if and only if $f'(x_\beta) \neq 0$. In particular, $f(L_{(\alpha)}) = f'(L_{(\alpha)}) = 0$.

Equation (4.1) also gives $f'(L_{-\alpha}) = 0$. We show furthermore that $f(L_{-\alpha}) = 0$. If $x \in L_{-\alpha}$, then $f(x) \in T$ by weights. However, for any $\beta \notin (\alpha)$, $[f(x), L_\beta] \subseteq [x, f(L_\beta)] + f'(L_{-\alpha+\beta}) = 0$, so $f(x) = 0$ by Lemma 4.3. Note also, that this implies $f(L) \cap T = 0$. In addition, we now have, for any $\beta \neq 0$, and $x \in L_\beta$, that $f(x) \neq 0$ if and only if $f'(x) \neq 0$.

Let $H = T + L_{(\alpha)}$. Then H is a subalgebra of L and $f(L) \subseteq L_{(\alpha)}$.

Let $K = \text{Nullspace}(f)$. Since $f \in \text{QDer}_T(L)_\alpha$, K is T -invariant, whence K is the direct sum of T weight spaces. We claim that K is a subalgebra of L . For this, take $x \in K_\beta$, $y \in K_\gamma$ with $\beta \neq 0$, $\gamma \neq 0$. If $\beta + \gamma = 0$, then $[x, y] \in T \subseteq K$. If $\beta + \gamma \neq 0$, we have $f'[x, y] = [f(x), y] + [x, f(y)] = 0$, so $f[x, y] = 0$, and again, $[x, y] \in K$.

Since $K \supseteq L_{(\alpha)}$, we have $L = H + K$.

Next, since $[L_{(\alpha)}, L_{(\alpha)}] \subseteq L_{(\alpha)}$,

$$[f(L), L_{(\alpha)}] = [f(L_{(\alpha)}), L_{(\alpha)}] \subseteq [L_{(\alpha)}, f(L_{(\alpha)})] + f'(L_{(\alpha)}) = 0.$$

Finally, since L is generated by special weight spaces, we need only show that these weight spaces are in the subalgebra K . For this, all that remains is to show that $f(L_\gamma) = 0$ if $\gamma \in (\alpha)$ is special. If $x \in L_\gamma$, then $f(x) \in L_{(\alpha)} = L_{(\gamma)}$ and $[f(x), L_{(\gamma)}] = [f(x), L_{(\alpha)}] = 0$. But then, by Lemma 4.10, $f(x) \neq 0$ would contradict the specialness of γ . ■

As a result of Lemma 4.2, Theorem 4.5, and Lemma 4.11, we have the following.

THEOREM 4.12. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$ a torus with $T = L_0$, and $Z(L) = 0$. Suppose L is generated by special weight spaces. Then*

$$\text{QDer}(L) = \text{Der}(L) \oplus (I_L).$$

COROLLARY 4.13. *If L is a parabolic subalgebra of a split, simple Lie algebra of rank > 1 over a field of characteristic 0, then*

$$\text{QDer}(L) = \text{ad}(L) \oplus (I_L).$$

Proof. The parabolic subalgebras are complete [11, 20]. ■

COROLLARY 4.14. *Let L be a Lie algebra of dimension > 3 in characteristic > 3 . Suppose further that $C(L) = (I_L)$ and that L has a nonsingular trace form on some representation. Then*

$$\text{QDer}(L) = \text{ad}(L) \oplus (I_L).$$

Proof. (We outline the underlying ideas and refer the reader to [19, Chap. II, V] for necessary background details.) We may assume the base field is algebraically closed. The trace-form assumption implies L is classical [19, p. 28] (this uses the assumption characteristic > 3) and the fact that $C(L) = (I_L)$ then implies L is simple. By a theorem of Kaplansky [10, Theorem 3], $\dim(L) > 3$ implies $\text{rank}(L) > 1$ for such algebras and so the simplicity of L implies all roots are special. Finally, Block [2] has shown that $\text{Der}(L) = \text{ad}(L)$ for these algebras. ■

Remark 4.15. Corollary 4.14 is an extension of the result of Hopkins [7] that these algebras have no nontrivial *antiderivations*, that is, $f \in \text{Hom}(L, L)$ such that $(f, f, -f) \in \Delta(L)$. This follows from the above since $\text{Der}(L) + (I_L)$ contains no antiderivations in characteristic > 3 .

COROLLARY 4.16. *Let L be a Lie algebra as in Theorem 4.12 except that L need not be indecomposable. Then*

$$\text{QDer}(L) = \text{Der}(L) \oplus C(L), \text{ direct sum as Lie algebras.}$$

Proof. By Lemma 3.4 and Theorem 4.12, we have $\text{QDer}(L) \subseteq \text{Der}(L) + Z_{C(L)}(\text{QDer}(L)) \subseteq \text{Der}(L) + C(L) \subseteq \text{QDer}(L)$. Since it is clear that $\text{Der}(L) \cap C(L) = 0$ for centerless Lie algebras, the result follows. ■

Remark 4.17. Corollary 4.16 shows that for such a Lie algebra, L , $C(L) \subseteq Z_{\text{QDer}(L)}(\text{Der}(L))$. Example 5.7 shows that this is not the case for general centerless Lie algebras.

Let $L = T + N$, with T a torus, N a nilpotent ideal, $\dim(T) = \dim(N/[N, N])$, and let U be a T -invariant complement of $[N, N]$ in N . Write $U = \bigoplus_{\alpha} U_{\alpha}$, where the U_{α} are one-dimensional weight spaces for T . It is well known that one can associate a graph $\Gamma(T, N)$ to $T + N$ as follows: Vertices of $\Gamma(T, N)$ are the weights of T on U while an edge joins the weights α and β if and only if $\alpha + \beta$ is a weight. Since U is isomorphic to $N/[N, N]$ as a T -module, $\Gamma(T, N)$ does not depend on the choice of U . (See, for example, [12] where we associate Lie algebras to arbitrary graphs.)

COROLLARY 4.18. *Suppose that $L = T + N$ is a semi-direct sum with T a torus, N a nilpotent ideal, $\dim(T) = \dim(N/[N, N])$, and $Z(L) = 0$. Suppose, further, that the weights of T in $N/[N, N]$ are disjoint from those in $[N, N]$ and that $\Gamma(T, N)$ has no isolated vertices. Then,*

$$\text{QDer}(L) = \text{ad}(L) \oplus C(L).$$

If $\Gamma(T, N)$ is connected, then

$$\text{QDer}(L) = \text{ad}(L) \oplus (I_L).$$

Proof. Since $Z(L) = 0$, and $\Gamma(T, N)$ has no isolated vertices, $L = T + N$ is complete, so that $\text{Der}(L) = \text{ad}(L)$ (see [11] or [12]). Let U be a subspace of N such that $N = U + [N, N]$. Then U generates N , and the weights of T on U are special, so Corollary 4.16 gives the first statement. If $\Gamma(T, N)$ is connected, then L is indecomposable and the second statement follows from Theorem 4.12. ■

The condition in Corollary 4.18 that the weights in $N/[N, N]$ and $[N, N]$ are disjoint is superfluous in characteristic 0 since it is already implied by the other hypotheses.

As a typical application, we get the following.

COROLLARY 4.19. *Let N be a nonabelian free-nilpotent Lie algebra, of dimension > 1 , over a field of characteristic p , i.e., $N = F/F^n$ where F is free and $n > 2$. Let T be a maximal torus of $\text{Der}(N)$, and let L be the semidirect sum $T + N$. If either $p = 0$, or $p \geq n$,*

$$\text{QDer}(L) = \text{ad}(L) \oplus (I_L).$$

Remark 4.20. The conclusion of Corollary 4.19 fails for $\dim(N) = 1$. In that case, L is the two-dimensional, nonabelian Lie algebra and $\text{QDer}(L) = \mathfrak{gl}(L)$ (Theorem 3.8).

5. QUASI-CENTROIDS OF LIE ALGEBRAS

5.1. Preliminary Remarks

The centroid $C(L)$ is an associative algebra and for centerless L it is commutative. As we note below, $\text{QC}(L)$ is not closed under composition in general. Indeed, one of our main results is that $\text{QC}(L)$ is a commutative algebra if L is centerless.

First we have the following.

LEMMA 5.1. $[C(L), \text{QC}(L)] \subseteq \text{Hom}(L, Z(L))$. Thus, if $Z(L) = 0$, $C(L)$ centralizes $\text{QC}(L)$.

Proof. Let $f \in C(L)$, $g \in \text{QC}(L)$, $x, y \in L$. Then $[[f, g](x), y] = [f(g(x)), y] - [g(f(x)), y] = f([g(x), y]) - [f(x), g(y)] = f([g(x), y]) - f([x, g(y)]) = 0$. ■

Note that $\text{QC}(L)$ is a Jordan algebra, using the operation $f_1 \bullet f_2 = (f_1 f_2 + f_2 f_1)/2$ for any elements $f_1, f_2 \in \text{QC}(L)$. It follows that $\text{QC}(L)$ is a Lie algebra with the operation $[f_1, f_2] = f_1 f_2 - f_2 f_1$ if and only if $\text{QC}(L)$ is also an associative algebra (with respect to composition). We shall show, for centerless L , that $\text{QC}(L)$ is commutative and so, in particular, is closed under composition. However, it need not be closed under composition otherwise, not even for $\dim(Z(L)) = 1$, as the following example shows.

EXAMPLE 5.2. Let N be the Heisenberg algebra, of dimension $2n + 1$. Then, with respect to the usual basis (see Section 2), $QC(N)$ has the form

$$\begin{pmatrix} A & B & 0 \\ C & A^t & 0 \\ e & f & g \end{pmatrix},$$

where A, B, C are $n \times n$ matrices, with B, C skew symmetric, and e, f, g are field elements. Hence, $QC(L)$ is not closed under composition.

Remark 5.3. Note by Lemma 3.1(3) that we may regard $QC(L)$ as an L -submodule of $\text{Hom}(L, L)$.

LEMMA 5.4. (1) For $x \in L$, $f \in QC(L)$, $[x, f(x)] = 0$ and so $\text{ad}(x)$ and $\text{ad}(f(x))$ commute.

(2) Let $f \in QC(L)$, $x, y \in L$. Then $(x \cdot f)(y) = -(y \cdot f)(x)$.

(3) Let $f \in QC(L)$, $x \in L$. Then $(\text{ad}(f(x)))^m = (\text{ad}(x))^m \circ f^m$ for all $m \geq 0$.

Proof. (1) and (2) are easy verifications, while (3) follows by induction on m using (1). ■

LEMMA 5.5. (1) If $f \in C(L)$, then $\text{Ker}(f)$ and $\text{Im}(f)$ are ideals in L .

(2) If L is indecomposable, and if $0 \neq f \in C(L)$, is such that x^2 does not divide the minimal polynomial of f , then f is invertible.

(3) If L is indecomposable and $C(L)$ consists of semisimple elements then $C(L)$ is a field.

Proof. (1) is clear. (2) follows from (1), since the minimal-polynomial hypothesis forces $L = \text{ker}(f) \oplus \text{Im}(f)$. (3) follows from (2). ■

5.2. On Commutativity of $QC(L)$

We show that $QC(L)$ is commutative for centerless L .

Notation. Recall that a Cartan subalgebra, H , of a Lie algebra, L , is a nilpotent subalgebra of L which equals its normalizer. Also, $L = L_0 \oplus L_1$, where L_0 is the Fitting null-component, and $L_1 = \sum_{x \in H} L_{1,x}$ is the Fitting 1-component of L with respect to $\text{ad}(H)$, where $L_{1,x}$ is the Fitting 1-component of L with respect to $\text{ad}(x)$ (see [9, p. 39 et seq]). Further, H is a Cartan subalgebra of L if and only if H is nilpotent and is the Fitting null-component of $\text{ad}(H)$ (see [9, p. 57]).

Also, H acts on $QC(L)$, (Remark 5.3), and we let $QC(L)_0, QC(L)_1$ denote the Fitting null- and 1-component of $QC(L)$ under this action.

First we show the following.

LEMMA 5.6. *Let H be a Cartan subalgebra of L . If $f \in \text{QC}(L)$, then $f(H) \subseteq H$.*

Proof. By Lemma 5.4, (1) and (3), for all $j \geq 0$, $y \in L$,

$$(\text{ad}(y))^{j+1} \circ f = (\text{ad}(y))^j \circ \text{ad}(f(y)) = \text{ad}(f(y)) \circ (\text{ad}(y))^j,$$

so that f preserves the Fitting null-component of $\text{ad}(y)$ for all $y \in H$. ■

Benoist states the above result for the case of semisimple Lie algebras over fields of characteristic 0 [1, p. 903].

EXAMPLE 5.7. It is not necessarily the case that $\text{QC}(L)$ preserves L_1 . A counterexample, useful for other purposes, is as follows: Let L have a basis x_0, \dots, x_5 , with

$$\begin{aligned} [x_0, x_1] &= x_1, & [x_0, x_3] &= x_3, & [x_0, x_5] &= x_5, \\ [x_1, x_2] &= x_5, & [x_3, x_4] &= x_5, \end{aligned}$$

and with other products 0. Then $C(L)$ is spanned by I_L and f_1, f_2 , where $f_1(x_0) = x_2$, $f_1(x_1) = -x_5$, $f_2(x_0) = x_4$, $f_2(x_3) = -x_5$, while otherwise, $f_i(x_j) = 0$. And $\text{QC}(L)$ is spanned by $C(L)$ and f_3 , where $f_3(x_1) = -x_4$, $f_3(x_3) = x_2$ with $f_3(x_i) = 0$ for $i \neq 1, 3$. A Cartan subalgebra, H , is spanned by x_0, x_2, x_4 . With respect to this Cartan subalgebra, L_1 is spanned by x_1, x_3, x_5 , but $f_3(L_1) \not\subseteq L_1$.

Referring back to Remark 4.17, we note that L has a derivation d such that $d(x_2) = x_4$, $d(x_3) = x_1$, while $d(x_i) = 0$ if $i \neq 2$ or 3 . Note that $f_1 \circ d \neq d \circ f_1$, so that $C(L) \neq Z_{C(L)}(\text{Der}(L))$.

In view of the next lemma, it is also worth noting that $f_3(H) = 0$ and $f_3(L_1) \subseteq Z(H)$.

LEMMA 5.8. *Let H be a Cartan subalgebra of L and let $H \oplus L_1$ be the Fitting decomposition of L for the action of $\text{ad}(H)$ ($H = L_0$). Let $\text{QC}(L)_0 + \text{QC}(L)_1$ be the Fitting decomposition of $\text{QC}(L)$ regarded as an H -module (see Remark 5.3). Then*

- (1) $\text{QC}(L)_0(L_1) \subseteq L_1$.
- (2) $\text{QC}(L)_1(H) = 0$.
- (3) $\text{QC}(L)_1(L_1) \subseteq Z(H)$.
- (4) $(H \cdot \text{QC}(L)_0)(L_1) = 0$.
- (5) $L_1 \cdot \text{QC}(L)_0 = 0$.

Proof. The map $\text{QC}(L) \otimes L \rightarrow L$ such that $f \otimes x \mapsto f(x)$ is an L -map, while, for L -modules M, N , one has $M_0 \otimes N_1 + M_1 \otimes N_0 \subseteq (M \otimes N)_1$. Thus (1) holds and $\text{QC}(L)_1(H) \subseteq L_1$. Then, by Lemma 5.6, $\text{QC}(L)_1(H) \subseteq L_1 \cap H = 0$, which is (2).

(3) If $f \in \text{QC}(L)_1$, then by (2) $[H, f(L_1)] = [f(H), L_1] = 0$. Thus $f(L_1) \subseteq L_0 = H$ and also $f(L_1) \subseteq Z(H)$.

(4) Using (2) of Lemma 5.4 and (2) of this lemma,

$$(H \cdot \text{QC}(L)_0)(L_1) = (L_1 \cdot \text{QC}(L)_0)(H) \subseteq \text{QC}(L)_1(H) = 0.$$

(5) Suppose that $L_1 \cdot \text{QC}(L)_0 \neq 0$. Then, by the proof of (4), $(L_1 \cdot \text{QC}(L)_0)(L_1) \neq 0$ and then, noting that $L_1 \cdot \text{QC}(L)_0 \subseteq \text{QC}(L)_1$, (3) implies that $[(L_1 \cdot \text{QC}(L)_0)(L_1), L_1] \neq 0$. Hence there exist $x, y, z \in L_1, f \in \text{QC}(L)_0$ such that $[(x \cdot f)(y), z] \neq 0$. We may assume that the ground field is algebraically closed and so, without loss of generality, that x, y, z are in weight spaces for H . Say $x \in L_\alpha, y \in L_\beta, z \in L_\gamma$ for nonzero α, β, γ . Since, by (3), $0 \neq (x \cdot f)(y) \in H$ it follows that $\alpha + \beta = 0$. However, using (2) of Lemma 5.4, and the QC-property

$$0 \neq [(x \cdot f)(y), z] = [y, (x \cdot f)(z)] = -[y, (z \cdot f)(x)] = -[(z \cdot f)(y), x],$$

from which it follows, similarly, that $\alpha + \gamma = 0$ and $\beta + \gamma = 0$, which yields the contradiction $\alpha = \beta = \gamma = 0$. ■

LEMMA 5.9. *If $Z(L) = 0$ then $\text{QC}(L)_0 = C(L)$.*

Proof: Since $C(L) = \{f \in \text{Hom}(L, L) \mid L \cdot f = 0\}$, we have $C(L) \subseteq \text{QC}(L)_0$. We must show that $L \cdot \text{QC}(L)_0 = 0$. By Lemma 5.8(5), it suffices to show that $(H \cdot \text{QC}(L)_0)(H) = 0$. But, by the quasicentroid property and Lemma 5.8(4),

$$[(H \cdot \text{QC}(L)_0)(H), L_1] = [H, (H \cdot \text{QC}(L)_0)(L_1)] = 0.$$

Thus $H \cdot \text{QC}(L)_0(H) \subseteq Z(L) = 0$. ■

LEMMA 5.10. *If $Z(L) = 0$, then $L_1 \cdot \text{QC}(L)_1 \subseteq \text{QC}(L)_0$.*

Proof. Suppose there exists $x \in L_1, f \in \text{QC}(L)_1$ such that $x \cdot f \notin \text{QC}(L)_0$. We may assume the base field algebraically closed and so, without loss of generality, assume that x and f lie in weight spaces for H . Take $x \in L_\alpha, f \in \text{QC}(L)_\beta$ for nonzero α, β ; then, since $x \cdot f \notin \text{QC}(L)_0$, we have $\alpha + \beta \neq 0$. Thus, since $(x \cdot f)(H) = 0$ by Lemma 5.8(2), there exists $y \in L_\gamma$, with $\gamma \neq 0$ such that $(x \cdot f)(y) \neq 0$. By Lemma 5.8(3), $\alpha + \beta + \gamma = 0$. Also, there exists $z \in L_\delta$, with $\delta \neq 0$, such that $[(x \cdot f)(y), z] \neq 0$ or else, by Lemma 5.8(3), $(x \cdot f)(y) \in Z(L) = 0$. By Lemma 5.4(2) and the QC property,

$$0 \neq [(x \cdot f)(y), z] = [y, (x \cdot f)(z)] = -[y, (z \cdot f)(x)] = -[(z \cdot f)(y), x].$$

From $(x \cdot f)(z) \neq 0$, we conclude $\alpha + \beta + \delta = 0$, so that $\gamma = \delta$. Now $(z \cdot f) \in \text{QC}(L)_1$, else $\delta + \beta = 0$, which would imply $\alpha = 0$. Then

$(z \cdot f)(y) \neq 0$ yields $\delta + \gamma + \beta = 0$ by Lemma 5.8(3). Hence $\alpha = \gamma = \delta$ and $\beta = -2\alpha$. Since $\alpha + \beta \neq 0$, $f(y) = f(z) = 0$, again, by Lemma 5.8(3). Then $(x \cdot f)(y) = [x, f(y)] - f([x, y]) = -f([x, y])$ so

$$0 \neq [(x \cdot f)(y), z] = -[f([x, y]), z] = -[[x, y], f(z)] = 0,$$

which is a contradiction. ■

THEOREM 5.11. *If L is a Lie algebra with $Z(L) = 0$, then $QC(L)$ is a commutative, associative algebra.*

Proof. We may assume the base field algebraically closed. Let $QC(L)_0$ and $QC(L)_1$ be as above. By Lemma 5.9 and Proposition 5.1, we have $[QC(L)_0, QC(L)] = [C(L), QC(L)] = 0$. Now it follows from Lemma 5.8(2) and (3) that $QC(L)_1 \circ QC(L)_1 = 0$, so that $[QC(L)_1, QC(L)_1] = 0$. ■

We have actually proved more about the structure of $QC(L)$.

THEOREM 5.12. *If L is a Lie algebra with $Z(L) = 0$, then $QC(L) = C(L) \oplus A$, with $C(L) \circ A \subseteq A$ and $A \circ A = 0$.*

Proof. In the above discussion, $A = QC(L)_1$. That $A \circ A = 0$ is shown in the proof of Theorem 5.11. ■

5.3. The Quasicentroid, the Radical, and the Maximal Nilpotent Ideal

This section is concerned, primarily, with the invariance of classically important ideals of L under the action of $QC(L)$. Example 5.7 shows that the derived algebra of L need not be invariant under $QC(L)$, even if L is centerless. However, we show that $\text{Rad}(L)$ and $\text{NilRad}(L)$ are invariant under $QC(L)$, and further, that the image of a nilpotent QC is always contained in the nilradical.

THEOREM 5.13. *$QC(L)$ preserves $\text{NilRad}(L)$.*

Proof. Take $f \in QC(L)$, $x \in \text{NilRad}(L)$. Then $f(x) \in \text{Tr}(L, \text{NilRad}(L))$, so that $K = (f(x)) + \text{NilRad}(L)$ is an ideal in L . Since $\text{ad}(x)$ is nilpotent, it follows from (3) of Lemma 5.4, that $\text{ad}(f(x))$ is nilpotent, which implies that K is nilpotent. Hence $K = \text{NilRad}(L)$ whence $f(x) \in \text{NilRad}(L)$. ■

The invariance of the radical depends on the following easy observation.

LEMMA 5.14. *If L is a Lie algebra, then $\text{Tr}(L, \text{Rad}(L)) = \text{Rad}(L)$.*

Proof. $[\text{Tr}(L, \text{Rad}(L)), \text{Tr}(L, \text{Rad}(L))] \subseteq \text{Rad}(L)$. ■

This yields a result about a class of generalized derivations slightly larger than $QC(L)$.

LEMMA 5.15. *Let L be a Lie algebra. If $(f, f', 0) \in \Delta(L)$, then*

$$f(\text{Rad}(L)) \subseteq \text{Rad}(L).$$

Proof. Since $[f(\text{Rad}(L)), L] = [\text{Rad}(L), f'(L)] \subseteq \text{Rad}(L)$, $f(\text{Rad}(L)) \subseteq \text{Tr}(L, \text{Rad}(L)) = \text{Rad}(L)$. ■

Our investigation of the image of nilpotent QCs is motivated by the following observation about the centroid.

LEMMA 5.16. *Let L be a Lie algebra, $f \in C(L)$ such that $f^m = 0$ with $m \geq 0$. Then $f(L)$ is an ideal of L and satisfies $f(L)^m = 0$. In particular, $f(L) \subseteq \text{NilRad}(L)$.*

Proof. As noted in Lemma 5.5, $f(L)$ is an ideal. The centroidal property of f also implies $f(L)^m = f^m(L^m) = 0$. ■

EXAMPLE 5.17. It is easy to construct centroidal elements of arbitrary nilpotency index m even in centerless algebras: Let L be a Lie algebra over κ with $Z(L) = 0$. For any $m \geq 0$, let $L\{m\} := L[\kappa[t]/(t^m)] (= L \otimes \kappa[t]/(t^m))$ where t is an indeterminant. Then $I \otimes$ (multiplication by t) is an element of $C(L\{m\})$ which is nilpotent of index m . This construction with L simple is a mainstay of [1].

The image of $f \in \text{QC}(L)$ need not be an ideal (see f_3 in Example 5.7). However, we still have the following.

THEOREM 5.18. *Let L be a Lie algebra, f a nilpotent element of $\text{QC}(L)$. Then $\langle f(L) \rangle$ is nilpotent, and so $f(L) \subseteq \text{NilRad}(L)$.*

For the proof of Theorem 5.18, we will need only the case $m = 2$ of the following theorem. (We remind the reader of our overall hypothesis that characteristic $\neq 2$.) However, the full statement of Theorem 5.19 presents an interesting analogue to Lemma 5.16.

THEOREM 5.19. *Let L be a Lie algebra over a field, κ , of characteristic p . Suppose $f \in \text{QC}(L)$ satisfies $f^m = 0$ with $m > 0$. If either $p = 0$ or $0 < m < p$, then $\langle f(L) \rangle^m = 0$.*

Notation. We represent by $[x_1, x_2, \dots, x_n]$ the left-associated monomial

$$[[\dots [[x_1, x_2], x_3], \dots], x_n].$$

Proof. We need only show that, for all n , and $x_i \in L$, $1 \leq i \leq n$, $[x_1, x_2, \dots, x_n] = 0$, whenever $x_i \in f(L)$ for at least m indices i . In turn, it suffices to show that $[f^{e_1}(x_1), f^{e_2}(x_2), \dots, f^{e_n}(x_n)] = 0$ if $\sum_{i=1}^n e_i \geq m$. Further, since $[f^{e_1}(x_1), f^{e_2}(x_2)] = [x_1, f^{e_1+e_2}(x_2)]$, we may assume $e_1 = 0$. Thus, the goal is to prove, for $\sum_{i=2}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), \dots, f^{e_n}(x_n)] = 0. \quad (5.1)$$

We prove (5.1) by induction on n .

If $e_2 \geq m$, then $[x_1, f^{e_2}(x_2)] = 0$; so (5.1) holds for $n = 2$.

Let $n > 2$ and suppose (5.1) holds for products of length $< n$.

For convenience in what follows, we abbreviate the sequence $f^{e_4}(x_4), \dots, f^{e_n}(x_n)$ by Φ ; these $n - 3$ factors will never change in our manipulations. We also employ $\{\mathcal{I}\}$, $\{\mathcal{J}\}$, $\{\mathcal{S}\}$, $\{\mathcal{Q}\}$, and $\{N\}$ respectively, to indicate that an argument used the induction hypothesis, the Jacobi identity, skew-commutativity, the QC relation, or equation (N), respectively.

The case $e_3 = 0$ will be handled first. We claim that, for $e_2 + \sum_{i=4}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), x_3, \Phi] = 0. \quad (5.2)$$

Proof of (5.2). We have

$$\begin{aligned} [x_1, f^{e_2}(x_2), x_3, \Phi] &= [x_3, f^{e_2}(x_2), x_1, \Phi] + [[x_1, x_3], f^{e_2}(x_2), \Phi] && \{\mathcal{I}\} \\ &= [x_3, f^{e_2}(x_2), x_1, \Phi], && \{\mathcal{J}\} \end{aligned}$$

and

$$\begin{aligned} [x_1, f^{e_2}(x_2), x_3, \Phi] &= [f^{e_2}(x_1), x_2, x_3, \Phi] && \{\mathcal{Q}\} \\ &= -[x_2, f^{e_2}(x_1), x_3, \Phi]. && \{\mathcal{S}\} \end{aligned}$$

Applying these two relations,

$$\begin{aligned} [x_1, f^{e_2}(x_2), x_3, \Phi] &= -[x_2, f^{e_2}(x_1), x_3, \Phi] \\ &= -[x_3, f^{e_2}(x_1), x_2, \Phi] \\ &= [x_1, f^{e_2}(x_3), x_2, \Phi] \\ &= [x_2, f^{e_2}(x_3), x_1, \Phi] \\ &= -[x_3, f^{e_2}(x_2), x_1, \Phi] \\ &= -[x_1, f^{e_2}(x_2), x_3, \Phi], \end{aligned}$$

from which (5.2) follows.

We always have

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -[x_2, f^{e_2}(x_1), f^{e_3}(x_3), \Phi]. \quad \{\mathcal{Q}\}, \{\mathcal{S}\} \quad (5.3)$$

Next we claim, for $\sum_{i=2}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi]. \quad (5.4)$$

Proof of (5.4).

$$\begin{aligned} [x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] &= [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] + [f^{e_3}(x_3), f^{e_2}(x_2), x_1, \Phi] && \{\mathcal{J}\}, \{\mathcal{S}\} \\ &= [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] + [x_3, f^{e_2+e_3}(x_2), x_1, \Phi] && \{\mathcal{Q}\} \\ &= [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] + 0. && \{5.2\} \end{aligned}$$

And, for $\sum_{i=2}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -[x_3, f^{e_2}(x_2), f^{e_3}(x_1), \Phi]. \quad (5.5)$$

Proof of (5.5).

$$\begin{aligned}
 & [x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] \\
 &= [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] \quad \{5.4\} \\
 &= -[x_3, f^{e_3}(x_1), f^{e_2}(x_2), \Phi] \quad \{5.3\} \\
 &= -[x_3, f^{e_2}(x_2), f^{e_3}(x_1), \Phi]. \quad \{5.4\}
 \end{aligned}$$

And, for $\sum_{i=2}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -[x_1, f^{e_3}(x_2), f^{e_2}(x_3), \Phi]. \quad (5.6)$$

Proof of (5.6).

$$\begin{aligned}
 & [x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] \\
 &= -[x_3, f^{e_2}(x_2), f^{e_3}(x_1), \Phi] \quad \{5.5\} \\
 &= [x_2, f^{e_2}(x_3), f^{e_3}(x_1), \Phi] \quad \{5.3\} \\
 &= -[x_1, f^{e_2}(x_3), f^{e_3}(x_2), \Phi] \quad \{5.5\} \\
 &= -[x_1, f^{e_3}(x_2), f^{e_2}(x_3), \Phi]. \quad \{5.4\}
 \end{aligned}$$

Also, for $\sum_{i=2}^n e_i \geq m$ with $e_2 + e_3 \geq 1$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -e_2[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi]. \quad (5.7)$$

Proof of (5.7). Proof is by a (sub)induction on e_2 .

When $e_2 = 0$, we have, $[x_1, x_2, f^{e_3}(x_3), \Phi] = [[x_1, x_2], f^{e_3}(x_3), \Phi] = 0$ by the main induction (on n).

Assume $e_2 > 0$ and that the corresponding result holds for $e_2 - 1$. We have,

$$\begin{aligned}
 & [x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] \\
 &= [f(x_1), f^{e_2-1}(x_2), f^{e_3}(x_3), \Phi] \quad \{Q\} \\
 &= [f(x_1), f^{e_3}(x_3), f^{e_2-1}(x_2), \Phi] \\
 &\quad + [f^{e_3}(x_3), f^{e_2-1}(x_2), f(x_1), \Phi] \quad \{\mathcal{F}\}, \{\mathcal{S}\} \\
 &= [x_1, f^{e_3+1}(x_3), f^{e_2-1}(x_2), \Phi] \\
 &\quad + [x_3, f^{e_2+e_3-1}(x_2), f(x_1), \Phi] \quad \{@\} \\
 &= [x_1, f^{e_2-1}(x_2), f^{e_3+1}(x_3), \Phi] \\
 &\quad - [x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] \quad \{5.4\}, \{5.5\} \\
 &= -(e_2 - 1)[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] \\
 &\quad - [x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] \quad \{\mathcal{F}\} \\
 &= -e_2[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi].
 \end{aligned}$$

Hence (5.7) holds.

Finally, for $e_2 + e_3 \geq 1$,

$$\begin{aligned}
 -e_2[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] &= [x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] \quad \{5.7\} \\
 &= -[x_1, f^{e_3}(x_2), f^{e_2}(x_3), \Phi] \quad \{5.6\} \\
 &= e_3[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi]. \quad \{5.7\}
 \end{aligned}$$

Since either the characteristic, p , is 0 or $e_2 + e_3 \leq m < p$, this yields $[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = 0$, completing the proof of Theorem 5.19. ■

Proof of Theorem 5.18. Observing that $f(Z(L)) \subseteq Z(L)$, we see that f naturally induces $\bar{f} \in \text{QC}(L/Z(L))$ wherein $\bar{f}(x + Z(L)) = f(x) + Z(L)$. Since $\text{NilRad}(L)/Z(L) = \text{NilRad}(L/Z(L))$, $f(L) \subseteq \text{NilRad}(L)$ if and only if $\bar{f}(L/Z(L)) \subseteq \text{NilRad}(L/Z(L))$. Thus, we may assume that $Z(L) = 0$.

We have $f = f_0 + f_1$, with $f_0 \in C(L)$, $f_1 \in A$, as in Theorem 5.12. Say $f^m = 0$. Then $0 = (f_0 + f_1)^m = f_0^m + m f_0^{m-1} f_1$. Since $f_0^m \in C(L)$ and $m f_0^{m-1} f_1 \in A$, $f_0^m = 0$. By Lemma 5.16, $f_0(L) \subseteq \text{NilRad}(L)$. Since $f_1^2 = 0$, $f_1(L) \subseteq \text{NilRad}(L)$ by Theorem 5.19. ■

The next corollary follows immediately.

COROLLARY 5.20. *Semisimple Lie algebras have no non-zero nilpotent QCs.*

Corollary 5.20 also follows from the “ $\text{QC}(L) = C(L)$ ” theory of the next subsection and results of Melville [14].

5.4. Relations between $\text{QC}(L)$ and $C(L)$ for Centerless Lie Algebras

LEMMA 5.21. *Let $Z(L) = 0$ and $f \in \text{QC}(L)$ and suppose x^2 does not divide the minimal polynomial of f . Then $L = \text{Ker}(f) \oplus \text{Im}(f)$, a direct sum of ideals.*

Proof. The hypothesis on the minimal polynomial implies at least a vector direct sum $L = \text{Ker}(f) \oplus \text{Im}(f)$. Also, $[\text{Ker}(f), f(L)] = [f(\text{Ker}(f)), L] = 0$. Since $Z_L(\text{Im}(L)) \cap Z_L(\text{Ker}(L)) = Z(L) = 0$, we must have $\text{Ker}(f) = Z_L(\text{Im}(L))$ and $\text{Im}(f) = Z_L(\text{Ker}(L))$. It follows that $\text{Im}(L)$ and $\text{Ker}(L)$ are ideals. ■

COROLLARY 5.22. *If $f \in \text{QC}(L)$ is semisimple, and L is centerless, then $f \in Z_{C(L)}(\text{GenDer}(L))$.*

Proof. We may assume the base field is algebraically closed and, by Lemma 3.4, that L is directly indecomposable. Let λ be a characteristic root of f . By Lemma 5.21, $\text{Ker}(f - \lambda I_L) = L$ (since $\text{Ker}(f - \lambda I_L) \neq 0$). That is, $f = \lambda I_L$. ■

Our first instance of equality in Eq. (1.6) is given by the following.

THEOREM 5.23. *If L is a centerless Lie algebra for which $C(L) = (I_L)$, then $\text{QC}(L) = (I_L)$.*

Proof. We may assume that the base field accommodates a Cartan subalgebra for L . By Lemma 5.9, it suffices to show that $\text{QC}(L)_1 = 0$. (See Lemma 5.8 for notation.)

Claim: $L_1 \cdot \text{QC}(L)_1 = 0$. Suppose the contrary. Then, by Lemmas 5.9 and 5.10, there exist $x \in L_1$ and $f \in \text{QC}(L)_1$, such that $x \cdot f = I_L$. By Lemma 5.8, $f(H) = 0$ and $f(L_1) \subseteq H$, and so, by Theorem 5.19, $\langle f(L) \rangle$ is abelian. Then,

$$H = I_L(H) = (x \cdot f)(H) \subseteq [x, f(H)] + f(L) = f(L),$$

but a Cartan subalgebra of a nonabelian algebra cannot be contained in an abelian ideal, proving the claim.

Now, by Lemma 5.8, Lemma 5.4(2), and the claim,

$$\text{QC}(L)_1(L) = \text{QC}(L)_1(L_1) = (H \cdot \text{QC}(L)_1)(L_1) = (L_1 \cdot \text{QC}(L)_1)(H) = 0.$$

■

Note that $C(L) = (I_L)$ means that the centroid coincides with the base field; algebras with this property are called *central* [9, p. 291].

If all of the elements of the centroid are semisimple, we move to the algebraic closure of the base field and the centroid may be diagonalized. Then L is a direct sum of weight spaces for the centroid, each of which is an ideal satisfying the hypothesis of Theorem 5.23. So we have the following.

COROLLARY 5.24. *If L is a centerless Lie algebra over a perfect field such that every element of $C(L)$ is semisimple, then $\text{QC}(L) = C(L)$.*

The next lemma includes a key to proving equality in inclusion (1.4) for centerless Lie algebras.

LEMMA 5.25. *Suppose $(f, -f, 0), (f, f, f') \in \Delta(L)$. Then, $\forall x, y, z \in L$,*

$$\begin{aligned} [x, f([y, z])] &= [x, [f(y), z]] = [x, [y, f(z)]], \\ f'([x, [y, z]]) &= [x, f'([y, z])]. \end{aligned}$$

Proof. For convenience, let $f' = 2g$. We have

$$g([x, y]) = [f(x), y] = [x, f(y)]. \quad (5.8)$$

Observe first that, $\forall x, y, z \in L$,

$$\begin{aligned} [g([x, y]), z] &= [[f(x), y], z] \\ &= [[f(x), z], y] + [f(x), [y, z]] \\ &= [g([x, z]), y] + [x, f([y, z])], \end{aligned}$$

so that, $\forall x, y, z \in L$,

$$[g([x, y]), z] + [g([z, x]), y] = [x, f([y, z])] = g([x, [y, z]]). \quad (5.9)$$

Adding the three equations obtained from Eq. (5.9) by cyclically permuting x, y, z , and using Eq. (5.9), we get, $\forall x, y, z \in L$,

$$\begin{aligned} & 2([g([x, y]), z] + [g([z, x]), y] + [g([y, z]), x]) \\ & = g([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) = 0. \end{aligned}$$

Thus, $\forall x, y, z \in L$,

$$[g([x, y]), z] + [g([z, x]), y] = -[g([y, z]), x] = [x, g([y, z])]. \quad (5.10)$$

Comparing Eq. (5.10) with Eq. (5.9), we have $\forall x, y, z \in L$,

$$[x, f([y, z])] = [x, g([y, z])].$$

Using Eq. (5.8),

$$[x, f([y, z])] = [x, [f(y), z]] = [x, [y, f(z)]],$$

and

$$\begin{aligned} f'([x, [y, z]]) &= 2g([x, [y, z]]) = 2[x, f([y, z])] = 2[x, g([y, z])] \\ &= [x, f'([y, z])]. \end{aligned}$$

■

Thus we have the following.

PROPOSITION 5.26. *Suppose $(f, -f, 0), (f, f, f') \in \Delta(L)$.*

- (1) *If $Z(L) = 0$ then $f \in C(L)$.*
- (2) *If $L = [L, L]$ then $f' \in C(L)$.*

As an immediate consequence of Proposition 5.26(1), we get the following.

THEOREM 5.27. *If $Z(L) = 0$ then $QC(L) \cap QDer(L) = C(L)$.*

THEOREM 5.28. *If $Z(L) = 0$ and $L = [L, L]$, then $QC(L) = C(L)$.*

Proof. We may assume the base field sufficiently large for L to admit a Cartan subalgebra H .

Considering the action of L on $QC(L)$, let $N := \text{Tr}(QC(L), C(L))$. By Lemmas 5.9 and 5.10, $L_1 \subseteq N$. Hence $L/N = (H + N)/N$ is nilpotent. Then, since $[L/N, L/N] = L/N$, $L = N$. Thus, $QC(L)_1 = H \cdot QC(L)_1 \subseteq C(L) = QC(L)_0$, whence $QC(L)_1 = 0$. ■

DEFINITION 5.1. We say that a subalgebra J of L is *taut* if the intersection of J with any abelian ideal of L is 0.

Conditions that ensure tautness include any of the following:

- (1) $J \cap \text{NilRad}(L) = 0$.
- (2) $0 \neq x \in J \Rightarrow (\text{ad}(x))^2 \neq 0$.
- (3) J is a torus acting faithfully on L .

Note also, if L has a taut Cartan subalgebra, then $Z(L) = 0$.

THEOREM 5.29. *Let L be a Lie algebra with a taut Cartan subalgebra. Then $\text{QC}(L) = \text{C}(L)$.*

Proof. Let H be a taut Cartan subalgebra of L . Suppose $\text{QC}(L) \neq \text{C}(L)$, then there exists $f \neq 0$ in $\text{QC}(L)_1$. Then $f(L) \subseteq H$ and $f(H) = 0$ by (2) and (3) of Lemma 5.8, so that $f^2 = 0$. By Theorem 5.19, $\langle f(L) \rangle$ is abelian, so that H has a nonzero intersection with an abelian ideal, contradicting the tautness of H . ■

LEMMA 5.30. *Let L be a Lie algebra with a taut Cartan subalgebra. If $0 \neq f \in \text{QC}(L) = \text{C}(L)$, then f cannot be nilpotent. In fact, the minimal polynomial of f cannot be divisible by x^2 .*

Proof. Let H be a taut Cartan subalgebra of L . Suppose $f \in \text{QC}(L)$ has minimum polynomial $x^2P(x)$. Letting $g = fP(f) \in \text{C}(L)$, we have $g^2 = 0$. By Lemma 5.16, $g(L)$ is an abelian ideal. Since $g(H) \subseteq H$ (Lemma 5.10), $g(H) = 0$. Then $g([H, L]) = [g(H), L] = 0$. However $L = H + [L, H]$, so this implies $g = 0$, which is a contradiction. ■

COROLLARY 5.31. *Let L be a directly indecomposable Lie algebra over a perfect field with a taut Cartan subalgebra. Then $\text{QC}(L)$ is a field in $Z(\text{GenDer}(L))$.*

Proof. By Lemma 5.30, $\text{QC}(L)$ consists of semisimple elements. So, by Theorem 5.29 (or by Corollary 5.22), $\text{QC}(L) = \text{C}(L)$. By Lemma 5.5(3), $\text{C}(L)$ is a field. That $\text{C}(L)$ centralizes $\text{GenDer}(L)$ follows from Corollary 5.22. ■

A torus as in Section 4.2 (i.e., $L = T_0$ and $Z(L) = 0$) is a taut Cartan subalgebra. This, in addition to Proposition 3.3(1), allows us to bring the above results together with those of Section 4. For example, using Corollary 4.16 we have the following two theorems.

THEOREM 5.32. *Let L contain a torus, T , with $T = L_0$. Suppose that $Z(L) = 0$, and that L is generated by special weight spaces. Then*

$$\text{GenDer}(L) = \text{Der}(L) \oplus \text{C}(L),$$

a direct sum as Lie algebras.

THEOREM 5.33. *If L is a parabolic subalgebra of a split simple Lie algebra of rank > 1 over a field κ of characteristic 0, then*

$$\text{GenDer}(L) = \text{ad}(L) \oplus (I_L) \simeq L \oplus \kappa.$$

6. INVARIANCE OF THE RADICAL UNDER GENERALIZED DERIVATIONS

We prove the result in the section title for Lie algebras in characteristic 0. We shall need the following lemmas.

LEMMA 6.1. *Let L be a semisimple Lie algebra over a field of characteristic zero. If $(f, f, 0) \in \Delta(L)$, then $f = 0$.*

Proof. We may assume the algebra simple and the ground field, κ , algebraically closed. If $\text{rank}(L) > 1$, the result follows from Theorem 4.12. That $\Delta(\mathfrak{sl}_2(\kappa))$ has no nonzero elements of the form $(f, f, 0)$ is a simple computation. ■

LEMMA 6.2. *Let S be a subalgebra and K an ideal in L such that $L = S + K$ is a semidirect sum. Suppose $(f, f, f') \in \Delta(L)$ and $k \in K$. Define $h: S \rightarrow S$ by*

$$h(s) \equiv f([s, k]) - f'([s, k]) \pmod{K}$$

for $s \in S$. Then $(h, h, 0) \in \Delta(S)$.

Proof. By Lemma 3.9(1), $\delta f = (f' - f) \circ \mu \in B^2(L, L) \cap B^2(L, \mathring{L})$. Using Lemma 3.9(2), if $s, s' \in S$, $k \in K$, $[s, (f' - f)([s', k])] + [(f' - f)([s, k]), s'] \equiv 0 \pmod{K}$. ■

THEOREM 6.3. *Let L be a Lie algebra over a field of characteristic 0, and let K be an ideal in L such that L/K is simple. Then, for any $f \in \text{QDer}(L)$, $f(K) \subseteq K$.*

Proof. Suppose false and let $L = S + K$, with S simple, be a counterexample. Since K is self-transporting, any QC of L preserves K . Hence, by Proposition 3.3(1), there is some $(f, f, f') \in \Delta(L)$ with $f(K) \not\subseteq K$.

Let $P = \{(f, f, f') \mid (f, f, f') \in \Delta(L)\}$. Then P is an S -module via $s \cdot (f, f, f') = (s \cdot f, s \cdot f, s \cdot f')$. Let M denote the submodule $\{(f, f, f') \in \Delta(L) \mid f(K) \subseteq K\}$. Let N be an S -module complement to M in P , so that $P = M \oplus N$. By hypothesis, $N \neq 0$, so let $(0, 0, 0) \neq (f, f, f') \in N$.

By Lemmas 6.1 and 6.2, for all $s \in S$, $k \in K$,

$$f([s, k]) \equiv f'([s, k]) \equiv [s, f(k)] \pmod{K},$$

the latter equivalence by the quasiderivation property. In particular, for all $s \in S$, $(s \cdot f)(K) \subseteq K$, so that $s \cdot (f, f, f') \in M \cap N = 0$. Thus,

$$s \cdot f = s \cdot f' = 0.$$

One of many useful consequences is

$$[f(s), x] = [s, (f' - f)(x)],$$

for $s \in S$, $x \in L$.

Let $\bar{f}: S \rightarrow S$ be determined by $\bar{f}(s) \equiv f(s) \pmod{K}$. Since S is simple, and $S \cdot f = 0$, it follows by Schur's Lemma that $\bar{f} = cI_S$ for some scalar c . Reassigning $(f, f, f') = (f - cI_L, f - cI_L, f' - 2cI_L)$, we force

$$f(S) \subseteq K,$$

while retaining the properties $S \cdot f = S \cdot f' = 0$ and $f(K) \not\subseteq K$.

We define a nonascending sequence of subspaces

$$L = K_0 \supset K_1 \supset \cdots \supset K_m \supset K_{m+1} = K_{m+2} = \cdots$$

wherein, for $i > 0$, $K_i = \{k \in K \mid f(k) \in K_{i-1}\}$ and $m+1$ is the minimal integer such that $f(K_{m+1}) \subseteq K_{m+1}$. Note that $K_1 = K$. Since $S \cdot f = 0$, K_i is an S -submodule of L . Since f induces S -injections $K_i/K_{i+1} \rightarrow K_{i-1}/K_i$, for $i \geq 1$, we see inductively (on i) that $S \cdot K_i/K_{i+1} = K_i/K_{i+1}$. This implies, in particular, that $L^S \subseteq \bigcap_i K_i$.

We claim that, for all i, j ,

$$[K_i, K_j] \subseteq K_{i+j}.$$

Proof of claim. By induction on $i+j$. The statement is clear for $i+j=0$. Assume that $i+j \geq 1$ and $[K_{i'}, K_{j'}] \subseteq K_{i'+j'}$ for $0 \leq i'+j' < i+j$. We have to show $f([K_i, K_j]) \subseteq K_{i+j-1}$. Since, $f([K_i, K_j]^S) \subseteq L^S \subseteq K_{i+j-1}$, we need only show $[S, f([K_i, K_j])] \subseteq K_{i+j-1}$. We use

$$[S, f([K_i, K_j])] \subseteq [S, (f' - f)([K_i, K_j])] + [S, f'([K_i, K_j])],$$

and deal with each of the terms on the right. First,

$$[S, (f' - f)([K_i, K_j])] \subseteq [f(S), [K_i, K_j]] \subseteq [K_0, [K_i, K_j]].$$

According to whether $i=0$, or $i>0$, respectively, use $K_j \subseteq K_{j-1}$ or $K_i \subseteq K_{i-1}$, respectively, together with the induction hypothesis, to conclude $[K_i, K_j] \subseteq K_{i+j-1}$; but, again by the induction hypothesis, $[K_0, K_{i+j-1}] \subseteq K_{i+j-1}$. Second,

$$[S, f'([K_i, K_j])] \subseteq f'([K_i, K_j]) \subseteq [f(K_i), K_j] + [K_i, f(K_j)].$$

According to whether $i=0$ or $i>0$, respectively, use $f(K_i) \subseteq K_0$ and $K_j \subseteq K_{j-1}$, or $f(K_i) \subseteq K_{i-1}$, respectively, together with the induction hypothesis, to conclude $[f(K_i), K_j] \subseteq K_{i+j-1}$; similarly, $[K_i, f(K_j)] \subseteq K_{i+j-1}$. Thus, the claim has been established.

One consequence of the last claim is that K_i is an ideal in L for all i . In particular, K_{m+1} is an ideal stabilized by f . Clearly, L/K_{m+1} also stands as a counterexample to the theorem, so that we may assume $K_{m+1} = 0$. Consequently, $L^S = 0$.

Next, we claim that, for $i \geq 0$ and $k \in K_i$,

$$f'(k) \equiv f(k) \pmod{K_{i+1}}.$$

Proof of claim. Since, $[S, K_i] = K_i$, the observation

$$(f' - f)([s, k]) = [s, (f' - f)(k)] = [f(s), k] \in [K_1, K_i] \subseteq K_{i+1},$$

for $s \in S$, $k \in K_i$, establishes the claim.

Since $[K_i, K_j] \subseteq K_{i+j}$, for all i, j , we can make

$$\hat{L} = K_0/K_1 \oplus K_1/K_2 \oplus \cdots \oplus K_{m-1}/K_m$$

into a graded Lie algebra by a standard construction. Namely, for $x \in K_i$, $y \in K_j$,

$$[x + K_{i+1}, y + K_{j+1}] = [x, y] + K_{i+j+1}.$$

It is immediate that

$$\text{Rad}(\hat{L}) = K_1/K_2 \oplus K_2/K_3 \oplus \cdots \oplus K_{m-1}/K_m.$$

It is straightforward to verify that $(\hat{f}, \hat{f}, \hat{f}') \in \Delta(\hat{L})$, where, $\hat{f}(K_0/K_1) = \hat{f}'(K_0/K_1) = 0$ and, for $i > 0$ and $k \in K_i$,

$$\hat{f}(k + K_{i+1}) = f(k) + K_i, \quad \hat{f}'(k + K_{i+1}) = f'(k) + K_i.$$

However, for $k \in K_i$, $f(k) \equiv f'(k) \pmod{K_{i+1}}$. Hence, $\hat{f} = \hat{f}' \in \text{Der}(\hat{L})$, so that \hat{f} preserves the radical of \hat{L} . Equivalently, $f(K) \subseteq K$, contradicting our assumption about f . The theorem is proved. ■

The main result of this section is the following.

THEOREM 6.4. *Let L be a Lie algebra over a field of characteristic 0 and $f \in \text{GenDer}(L)$. Then f preserves $\text{Rad}(L)$.*

Proof. By Lemma 5.15, the result holds for $f \in \text{QC}(L)$. We may assume that $f \in \text{QDer}(L)$. If L is solvable, the result holds trivially. Otherwise, let K be a maximal proper ideal of L containing the radical, and let $(f, f, f') \in \Delta(L)$. By Theorem 6.3, f preserves K . Then $f'([K, K]) \subseteq K$ and so the restriction, $f|_K$, of f to K is in $\text{QDer}(K)$. We may assume, inductively (on the dimension), that $f|_K$ preserves $\text{Rad}(K)$. However $\text{Rad}(K) = \text{Rad}(L)$. ■

7. APPLICATIONS

If V is a vector space over a field, then a skew-symmetric, nonassociative algebra structure on V is an element, μ , of $\mathcal{M}(V) = \text{Hom}(V \wedge V, V)$. We regard $\mathcal{M}(V)$ as an affine algebraic variety. The set of $\mu \in \mathcal{M}(V)$ satisfying the Jacobi identity is the algebraic subvariety, $\mathcal{L}(V)$, of Lie algebra structures on V . We mention three actions of $\text{GL}(V)$ on $\mathcal{M}(V)$:

$$\textbf{Action 1: } f \cdot_1 \mu = f \circ \mu \circ (f^{-1} \wedge f^{-1})$$

$$\textbf{Action 2: } f \cdot_2 \mu = \mu \circ (f^{-1} \wedge I_V)$$

$$\textbf{Action 3: } f \cdot_3 \mu = f \circ \mu$$

for $f \in \text{GL}(V)$ and $\mu \in \mathcal{M}(V)$.

Action 1 leaves $\mathcal{L}(V)$ fixed. Indeed, if $\gamma = f \cdot_1 \mu$, then the Lie algebra (V, γ) is isomorphic to (V, μ) . If the orbit, $\text{GL}(V) \cdot_1 \mu$ is a Zariski open subset of $\mathcal{L}(V)$, then (V, μ) is called *rigid* (see, e.g., [4, 16]). It is well known that if $H^2(L, L) = 0$, then L is rigid (though the converse is false [17]).

Actions 2 and 3 do not always preserve $\mathcal{L}(V)$, so in these cases, the interesting questions seem to be:

- (1) When is $f \cdot_i \mu \in \mathcal{L}(V)$ for $\mu \in \mathcal{L}(V)$, $f \in \text{GL}(V)$, $i = 2, 3$?
- (2) If $f \cdot_i \mu \in \mathcal{L}(V)$, under what conditions is $f \cdot_i \mu$ isomorphic to μ ?

Action 2 will be the subject of Section 7.1, and Action 3 that of Section 7.2.

7.1. Projective Doubles of Lie Algebras

Given a Lie algebra (L, μ) , a *projective double* on (L, μ) is a Lie algebra (L, ρ) (i.e., the vector spaces are the same) such that

$$\text{ad}_\rho(L) \subseteq \text{Der}(L).$$

We call a projective double *inner* if $\text{ad}_\rho(L) \subseteq \text{ad}_\mu(L)$, and we call $f \in \text{Hom}(L, L)$ a *doubling* of (L, μ) provided (L, ρ_f) is a Lie algebra (and hence a projective double of L), where

$$\rho_f(x, y) = (f^{-1} \cdot_2 \mu)(x, y) = \mu(f(x), y)$$

for all $x, y \in L$. We denote by $\text{DB}(L)$ the set of doublings of L .

Any inner projective double, (L, ρ) , on (L, μ) gives rise to a doubling, f , of (L, μ) such that $\rho_f = \rho$. For this, one simply chooses a basis e_i of L and defines f by choosing $f(e_i)$ so that $\text{ad}_\mu(f(e_i)) = \text{ad}_\rho(e_i)$. If (L, μ) is centerless, this doubling, f , is actually unique. Note further, that, since ρ_f

must be skew symmetric, a doubling, f , of (L, μ) must be an element of $\text{QC}(L, \mu)$ so that

$$\text{C}(L) \subseteq \text{DB}(L) \subseteq \text{QC}(L).$$

The next result is essentially [8, Theorem 2].

THEOREM 7.1 [Ikeda]. *Let (L, ρ) be an inner projective double of a centerless Lie algebra, (L, μ) . Then there exists a unique doubling, f , of (L, μ) such that*

- (1) f is a homomorphism of (L, ρ) into (L, μ) .
- (2) f is in the centroid of (L, ρ) .
- (3) $\ker(f) = \text{Z}(L, \rho)$.

Proof. Choose the unique doubling, f , as discussed above. We show that f is a homomorphism $f: (L, \rho) \rightarrow (L, \mu)$.

$$\begin{aligned} \text{ad}_\mu(f \circ \rho(x, y)) &= \text{ad}_\rho(\rho(x, y)) \\ &= \text{ad}_\rho(x) \text{ad}_\rho(y) - \text{ad}_\rho(y) \text{ad}_\rho(x) \\ &= \text{ad}_\mu(f(x)) \text{ad}_\mu(f(y)) - \text{ad}_\mu(f(y)) \text{ad}_\mu(f(x)) \\ &= \text{ad}_\mu(\mu(f(x), f(y))). \end{aligned}$$

Since (L, μ) is centerless, f is a homomorphism.

Thus, for $x, y \in L$,

$$f \circ \rho(x, y) = \mu(f(x), f(y)) = \rho(f(x), y),$$

which is (2).

(3) is clear since (L, μ) is centerless. ■

LEMMA 7.2. *Let $L = (L, \mu)$ be centerless. If $f \in \text{DB}(L)$ is nonsingular, then $f \in \text{C}(L)$.*

Proof. $f(\mu(x, y)) = f(\mu(ff^{-1}(x), y)) = f(\rho_f(f^{-1}(x), y)) = \mu(x, f(y))$, the last equality by Theorem 7.1(1). ■

THEOREM 7.3. *Let L be a directly indecomposable Lie algebra with a taut Cartan subalgebra. Then every inner projective double of L is isomorphic to L .*

Proof. By Theorem 5.29, Lemma 5.30, and Lemma 5.5(2), any nonzero element of $\text{QC}(L)$ is invertible. The result follows by Lemma 7.2 and Theorem 7.1(1). ■

COROLLARY 7.4. *If L is a simple Lie algebra, then any nonabelian inner projective double of L is isomorphic to L .*

As another special case, we recover a result equivalent to a remark in [8, last paragraph].

COROLLARY 7.5 [Ikeda]. *If L is a parabolic subalgebra of a simple Lie algebra in characteristic 0, then every projective double of L is isomorphic to L .*

Proof. Such algebras are complete [11, 20]. ■

COROLLARY 7.6. *Let $L = T + N$ where T is a torus acting faithfully on N , N is a nilpotent ideal with $\dim(N/N^2) = \dim(T)$, and the weights of the T -module structure induced on N/N^2 are disjoint from the weights in $[N, N]$, and further, L is indecomposable. Then every nonabelian projective double of L is isomorphic to L .*

Proof. Such algebras are complete [11, Proposition 4.1]. ■

There are natural occurrences of the situation of Corollary 7.6 (in addition to Borel subalgebras of semisimple Lie algebras in characteristic 0, which are covered by Corollary 7.5), e.g., $T + N$ with N nonabelian free nilpotent either in characteristic 0 or when the characteristic exceeds the index of nilpotency of N ; see also [11, 12].

EXAMPLE 7.7. It is not the case in general, for centerless, indecomposable L with $C(L) = QC(L)$, that every inner projective double of L is isomorphic to L . To produce a counterexample, we note first that a doubling, f , yields an isomorphic projective double, (L, ρ_f) if and only if f is nonsingular. Also, by Corollary 5.22 and Lemma 5.5(2), if L is indecomposable, then a semisimple doubling is nonsingular. Thus, if L is centerless and indecomposable, then every projective double of L is isomorphic to L unless $C(L)$ contains nilpotent elements. On the other hand, if $0 \neq f \in C(L)$ with f nilpotent, then the projective double, (L, ρ_f) , induced by f is not isomorphic to L . So, for our counterexample, it suffices to display a centerless, indecomposable Lie algebra, L , with $C(L) = QC(L)$ while $C(L)$ contains nilpotent elements. Example 5.17 yields many such algebras but we need the following lemma.

Recall the construction of $L\{m\}$ in Example 5.17.

LEMMA 7.8. *Let L be a centerless indecomposable Lie algebra over κ . Then $L\{m\}$ is centerless and indecomposable for all $m \geq 0$.*

Proof. It is clear that $L\{m\}$ is centerless. We prove the indecomposability by induction on m . For $m = 0$, we have $L\{0\} \simeq L$. Assume the lemma is true for m . Suppose $L\{m+1\} = A \oplus B$, a nontrivial direct sum of ideals. The natural homomorphism $\kappa[t]/(t^{m+1}) \rightarrow \kappa[t]/(t^m)$ induces an epimorphism $\pi: L\{m+1\} \rightarrow L\{m\}$. Thus $L\{m\} = \pi(A) + \pi(B)$. Since $[\pi(A) \cap \pi(B), L\{m\}] = [\pi(A) \cap \pi(B), \pi(A) + \pi(B)] \subseteq [\pi(A), \pi(B)] =$

0, we have $\pi(A) \cap \pi(B) \subseteq Z(L\{m\}) = 0$. Thus, by the induction hypothesis, one of $\pi(A)$, $\pi(B)$ is 0. Without loss of generality, $\pi(A) = 0$ and so $\pi(B) = L\{m\}$. Then $A \subseteq L \otimes (t^m)$ and $B + L \otimes (t^m) = L\{m+1\}$. But then $[L\{m+1\}, A] = [B, A] = 0$, contradicting $Z(L\{m+1\}) = 0$. ■

Thus, if L is centerless and indecomposable, and $L = [L, L]$, then $L\{m\}$ has these properties and also, by Theorem 5.28, $C(L\{m\}) = QC(L\{m\})$. Now, $I \otimes$ (multiplication by t) is a nilpotent element of $C(L\{m\})$.

EXAMPLE 7.9. Returning to the inclusions, $C(L) \subseteq DB(L) \subseteq QC(L)$, we observe that Example 5.7 gives a centerless, directly indecomposable Lie algebra for which both inclusions are proper:

First, $f_3 \in DB(L) \setminus C(L)$. (Note that (L, ρ_{f_3}) is the direct sum of the Heisenberg algebra spanned by x_1, x_3, x_5 , with $[x_1, x_3] = x_5$ and a three-dimensional abelian algebra; the homomorphism induced by f_3 maps this to a two-dimensional abelian subalgebra of L spanned by x_2, x_4 .)

To show that the second inclusion can be proper, note, by Lemma 7.2, that it suffices to produce a nonsingular $f \in QC(L) \setminus C(L)$, and $I + f_3$ is such a QC.

From this last example, one sees also that $DB(L)$ need not be a subspace of $QC(L)$: f_3 and I_L are in $DB(L)$ but $I_L + f_3$ is not. However, we shall show (Corollary 7.14) that, for centerless L , $DB(L)$ is closed under multiplication (composition).

Notation. Let $f \in QC(L)$. Define $A_f, B_f, C_f \in C^3(L, L)$ and $h_f \in C^2(L, L)$ so that, for $x, y, z \in L$,

$$A_f(x, y, z) = [f(x), [f(y), z]] + [f(y), [f(z), x]] + [f(z), [f(x), y]],$$

$$B_f(x, y, z) = [x, [f(y), z]] + [y, [f(z), x]] + [z, [f(x), y]],$$

$$C_f(x, y, z) = [f(x), [y, z]] + [f(y), [z, x]] + [f(z), [x, y]],$$

$$h_f(x, y) = [f(x), f(y)].$$

Using the Jacobi identity and the centroid and quasicentroid properties, we see that the following holds.

LEMMA 7.10. (1) If $f \in C(L)$, then $A_f = B_f = C_f = 0$.

(2) If $f \in QC(L)$, then $B_{f^2} = -2A_f$ and $C_f = -2B_f$.

In light of the cohomological connections in the next subsection, it is interesting to note the following.

PROPOSITION 7.11. If $f \in QC(L)$ and the characteristic of the base field is either 0 or > 3 , then f is a doubling if and only if $h_f \in Z^2(L, L)$.

Proof. The Jacobi identity on the multiplication $\mu \circ (f \wedge I)$ is

$$A_f = 0,$$

while the cocycle condition applied to h_f is $B_{f^2} - C_{f^2} = 0$, or $-6A_f = 0$. ■

LEMMA 7.12. *Let $Z(L) = 0$ and $f \in \text{QC}(L)_1$ (notation as in Section 5.2). Then $B_f = 0$ implies $f = 0$.*

Proof. Since $f(H) = 0$ and $f(L_1) \subseteq Z(H)$, we have $0 = B_f(H, L_1, L_1) = [H, [f(L_1), L_1]]$. Since $[f(L_1), L_1] \subseteq L_1$, we have $[f(L_1), L_1] = 0$ by Lemma 5.8(3) and therefore $f(L_1) \subseteq Z(L) = 0$ by Lemma 5.8(2). ■

THEOREM 7.13. *Let L be a centerless Lie algebra and let $f \in \text{QC}(L)$. Then $f \in \text{DB}(L)$ if and only if $f^2 \in C(L)$.*

Proof. We may assume the base field is large enough (e.g., algebraically closed) to assure that L has a Cartan subalgebra. Suppose $f = f_0 + f_1$ with $f_0 \in \text{QC}(L)_0 = C(L)$, $f_1 \in \text{QC}(L)_1$. Since $\text{QC}(L)_1^2 = 0$, $f^2 = f_0^2 + 2f_0f_1$ and so $B_{2f_0f_1} = B_{f^2} = -2A_f$. Now, $f \in \text{DB}(L)$ if and only if $A_f = 0$ (proof of Proposition 7.11). But, since $f_0f_1 \in \text{QC}(L)_1$, $B_{2f_0f_1} = 0$ if and only if $2f_0f_1 = 0$ (Lemma 7.12), which holds if and only if $f^2 = f_0^2 \in C(L)$. ■

COROLLARY 7.14. *If $Z(L) = 0$ then $\text{DB}(L)$ is closed under composition.*

Proof. By Theorem 7.13 and commutativity of $\text{QC}(L)$ (Theorem 5.11). ■

7.2. Quasiderivations and Robustness

Let (L, μ) be a Lie algebra and f a nonsingular element of $\text{Hom}(L, L)$. Observe that $(L, f \circ \mu) \equiv (L, f \circ \mu)$ is a Lie algebra if and only if f satisfies: $\mu(x, f(\mu(y, z))) + \mu(y, f(\mu(z, x))) + \mu(z, f(\mu(x, y))) = 0$, for all $x, y, z \in L$.

DEFINITIONS. Let (L, μ) be a Lie algebra and f a nonsingular element of $\text{Hom}(L, L)$ such that $(L, f \circ \mu)$ is a Lie algebra. We call such $(L, f \circ \mu)$ a *perturbation* of (L, μ) , and the perturbation is said to be *inessential* if $f \circ \mu = c \circ \mu$ for some $c \in C(L)$. We say (L, μ) is *robust* if every perturbation of (L, μ) is inessential.

PROPOSITION 7.15. *An inessential perturbation of (L, μ) is necessarily isomorphic to (L, μ) .*

Proof. Suppose $f \circ \mu = c \circ \mu$, where f is nonsingular and $c \in C(L, \mu)$. It suffices to show that $f \circ \mu = c' \circ \mu$ with $c' \in C(L, \mu)$ and c' nonsingular, for such a c' induces an isomorphism from $(L, f \circ \mu)$ to (L, μ) .

Let $n = \dim(L)$ and $K = \text{Ker}(c^n)$. Since $f^n(\mu(K, L)) = c^n(\mu(K, L)) = \mu(c^n(K), L) = 0$, $K \subset Z(L, \mu)$ by the nonsingularity of f . The nonsingularity of f also gives $K \cap \mu(L, L) = 0$. Thus, we may choose $M \supset \mu(L, L)$ so that $L = K \oplus M$ (this is a direct sum of ideals for (L, μ)). With respect to this decomposition, let π be the projection of L to M . Define $c' \in \text{Hom}(L, L)$ so that $c'(m) = \pi(c(m))$ for $m \in M$ and $c'(k) = k$ for $k \in K$. Since c induces a nonsingular transformation $L/K \rightarrow L/K$, c' is a bijection. Since $c'(x) = c(x)$ for $x \in \mu(L, L)$ and $c'(x) \equiv c(x) \pmod{Z(L, \mu)}$ for all $x \in L$, we have $c' \in C(L, \mu)$. ■

Following [18], we define $\text{sq}^2: C^2(L, L) \rightarrow C^3(L, L)$ by

$$\text{sq}^2(\nu)(x, y, z) = \nu(x, \nu(y, z)) + \nu(y, \nu(z, x)) + \nu(z, \nu(x, y))$$

so that (L, ν) is a Lie algebra if and only if $\text{sq}^2(\nu) = 0$. If (L, μ) is a Lie algebra, then for $h \in \text{Hom}(L, L)$,

$$h \circ (\delta(h \circ \mu)) = \text{sq}^2(h \circ \mu),$$

where $\delta: C^2(L, L) \rightarrow C^3(L, L)$ is the coboundary map (computed with respect to $L = (L, \mu)$). Hence, if h is nonsingular, then $(L, h \circ \mu)$ is a Lie algebra if and only if $h \circ \mu \in Z^2(L, L)$. This leads to the following.

PROPOSITION 7.16. *Let $L = (L, \mu)$ be a Lie algebra over a field κ , with $|\kappa| > \dim(L)$. Then L is robust if and only if $Z^2(L, L) \cap B^2(L, \mathring{L}) = C(L) \circ \mu$.*

Proof. The sufficiency follows immediately from the above. Conversely, suppose (L, μ) is robust and $h \circ \mu \in Z^2(L, L)$, then, using the field hypothesis, we may choose $c \in \kappa$ such that $cI_L + h$ is nonsingular. Since $(cI_L + h) \circ \mu \in Z^2(L, L)$, $(cI_L + h) \circ \mu \in C(L) \circ \mu$ by robustness. Hence also $h \circ \mu \in C(L) \circ \mu$. ■

Thus, it is worth highlighting a cohomological condition that characterizes the collapsing of the second inclusion in Eq. (1.3).

PROPOSITION 7.17. $\text{QDer}(L) = \text{Der}(L) + C(L)$ if and only if $B^2(L, L) \cap B^2(L, \mathring{L}) = C(L) \circ \mu$.

Proof. Suppose $B^2(L, L) \cap B^2(L, \mathring{L}) = C(L) \circ \mu$. If $f \in \text{QDer}(L)$, then, by Lemma 3.9(1), $\delta f = \mathring{\delta}(f' - f) \in B^2(L, L) \cap B^2(L, \mathring{L})$, so that $(f' - f) \circ \mu = g \circ \mu$ with $g \in C(L)$. Then $f - g \in \text{Der}(L)$.

Conversely, suppose $\text{QDer}(L) = \text{Der}(L) + C(L)$. If, for $f, g \in \text{Hom}(L, L)$, $\delta f = \mathring{\delta}g \in B^2(L, L) \cap B^2(L, \mathring{L})$, then $(f, f, f + g) \in \Delta(L)$ by Lemma 3.9(1). Hence, $f = d + c$ with $d \in \text{Der}(L)$, $c \in C(L)$, so that $\delta f = c \circ \mu$. ■

COROLLARY 7.18. *If $H^2(L, L) = 0$ and if $\text{QDer}(L) = \text{Der}(L) + C(L)$, then L is robust.*

As a consequence, we have the following.

THEOREM 7.19. *If (L, μ) is a parabolic subalgebra of a split simple Lie algebra, of rank > 1 , over a field of characteristic 0, then L is robust.*

The cohomological condition in Corollary 7.18 indicates the existence of many algebras that are both robust and rigid (see also [11]). It is useful to illustrate the independence of these properties with the following examples.

EXAMPLE 7.20. The three-dimensional simple Lie algebra is an example of a rigid ($H^2(L, L) = 0$) but non-robust Lie algebra. It is indecomposable, which is of interest because decomposables are trivially nonrobust. The nonrobustness is easy to see by looking at the “nonsplit form” over the complexes, i.e., the “ $i - j - k$ ” vectors. Any diagonal h makes $(L, h \circ \mu)$ into a Lie algebra. Indeed, any symmetric matrix works.

Another example is given by the two-dimensional nonabelian Lie algebra.

EXAMPLE 7.21. A robust but nonrigid Lie algebra; a Lie algebra, L , for which $H^2(L, L) \neq 0$ and yet $Z^2(L, L) \cap B^2(L, \check{L}) = B^2(L, L) \cap B^2(L, \check{L}) = (\mu)$. The example is of the form $T + N$ where, T is a three-dimensional torus and N is seven-dimensional with one-dimensional weight spaces, the weights being $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$. Let x_1, x_2, x_3 be the generators of N corresponding to the simple weights and so that N has basis $\{x_1, x_2, x_3, [x_1, x_2], [x_2, x_3], [x_1, x_3], [x_1, [x_2, x_3]]\}$. We form the multiplication with $[x_1, x_2]$ central, so that

$$[x_2, [x_1, x_3]] = [x_1, [x_2, x_3]], \quad [x_3, [x_1, x_2]] = 0.$$

The nonrigidity of L is demonstrated by the one-parameter family of multiplications with the same basis and T -weight system, with multiplication in N satisfying

$$[x_2, [x_1, x_3]] = a[x_1, [x_2, x_3]], \quad [x_3, [x_1, x_2]] = (a - 1)[x_1, [x_2, x_3]],$$

where $a \in$ the base field (which we assume to be infinite). The members of the family with $a \neq 0, 1$ are not isomorphic to L (e.g., for $a \neq 0, 1$, $Z(N) = ([x_1, [x_2, x_3]])$).

EXAMPLE 7.22. Finally, a Lie algebra satisfying $B^2(L, L) \cap B^2(L, \check{L}) = (\mu)$ but $Z^2(L, L) \cap B^2(L, \check{L}) \neq (\mu)$ is given by

$$\begin{aligned} [x_1, x_3] &= x_3, & [x_1, x_5] &= x_5, & [x_1, x_6] &= x_6, \\ [x_1, x_7] &= 2x_7, & [x_2, x_4] &= x_4, & [x_2, x_5] &= x_5, \\ [x_2, x_6] &= 2x_6, & [x_2, x_7] &= x_7, & [x_3, x_4] &= x_5. \end{aligned}$$

The verification is facilitated by the following: that $\text{QDer}(L) = \text{Der}(L) + C(L)$ follows from Theorem 4.12 (the torus is spanned by x_1, x_2), hence, by Proposition 7.17, $B^2(L, L) \cap B^2(L, \mathring{L}) = (\mu)$, (μ is $[\cdot, \cdot]$); on the other hand, $(L, h \circ \mu)$ is a Lie algebra where $h(x_i) = x_i$, for $1 \leq i \leq 5$, and $h(x_6) = 2x_6$, $h(x_7) = 3x_7$, so that $Z^2(L, L) \cap B^2(L, \mathring{L}) \neq (\mu)$ by Proposition 7.16 and Corollary 7.18.

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